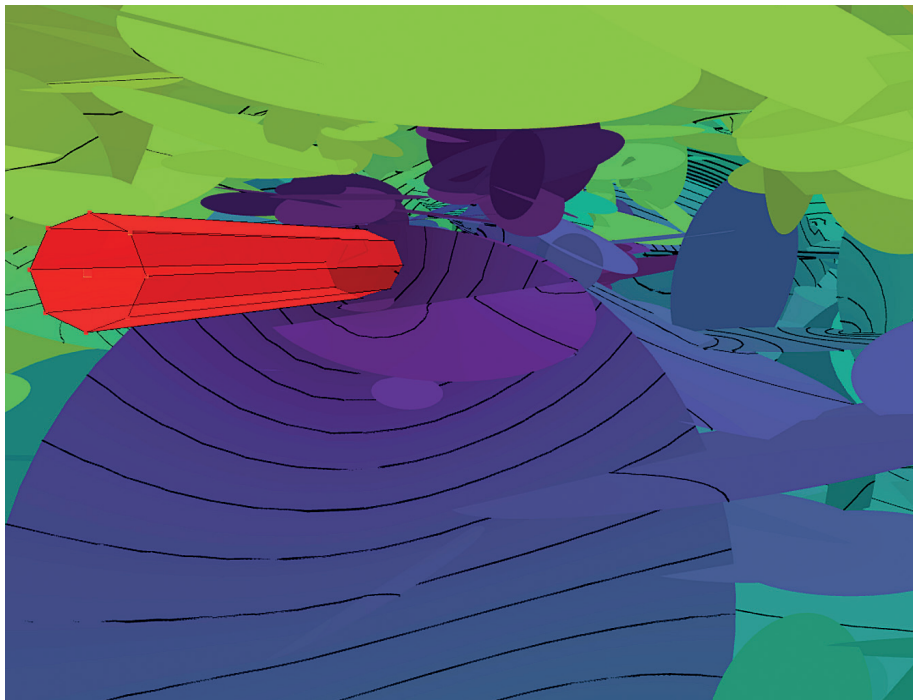




AN ANALYTIC MODEL FOR THREE- DIMENSIONAL GROUNDWATER FLOW IN DISCRETE FRACTURE NETWORKS

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Cover figure: Visualization generated with AnDFN, showing a tunnel (red) intersecting a discrete fracture network represented by circular planar fractures colored according to hydraulic head (Viridis colormap). Black contour lines indicate equipotential lines within the fracture network.

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**En analytisk modell för tredimensionellt
grundvattenflöde i diskreta spricknätverk**

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PREFACE

This report presents the development of a new analytic model for simulating groundwater flow in fractured crystalline rock. The work addresses a major challenge in hydrogeology and rock engineering, namely that most groundwater flow in hard rock occurs through fractures rather than in the surrounding rock matrix, making the flow highly heterogeneous and difficult to model accurately.

The report also contains some practical applications, particularly tunnel inflow analysis and hydraulic upscaling. In the tunnel case, the model is used to simulate inflow to a tunnel; across 50 realizations, the results show that tunnel inflow is spatially variable and strongly dependent on fracture connectivity, not only on fracture transmissivity.

The study is funded by BeFo, KTH, and WSP, and the engagement of the reference group consisting of Tomas Bym, Robert Earon, Åsa Fransson, Caroline Karlsson, Jessica Lindmark, Jan-Olof Selroos, Martin Stigsson, Björn Stille, Maryam Zamzami and Patrik Vidstrand is acknowledged.

FÖRORD

Denna rapport presenterar utvecklingen av ett nytt analytiskt ramverk för att simulera grundvattenflöde i sprickig berggrund. Arbetet har sin grund i en stor utmaning inom hydrogeologi och bergteknik, nämligen att det mesta av grundvattenflödet i berggrunden sker genom sprickor snarare än genom den omgivande bergmatrisen, vilket gör flödet mycket heterogent och svårt att modellera korrekt.

Rapporten innehåller också några praktiska tillämpningar, särskilt analys av tunnel-inläckage och hydraulisk uppskalning. I tunnelfallet används ramverket för att simulera inflöde till en tunnel; över 50 simuleringar visar resultaten att inläckaget varierar och är beroende av spricknätverket, inte bara av sprickornas transmissivitet.

Studien finansieras av BeFo, KTH och WSP och en referensgrupp bestående av Tomas Bym, Robert Earon, Åsa Fransson, Caroline Karlsson, Jessica Lindmark, Jan-Olof Selroos, Martin Stigsson, Björn Stille, Maryam Zamzami och Patrik Vidstrand har bidragit.

SUMMARY

This study presents the development of a new analytic model for simulating groundwater flow and transport in fractured crystalline rock. The work addresses a major challenge in hydrogeology and rock engineering, namely that most groundwater flow in hard rock occurs through interconnected fractures rather than through the surrounding rock matrix, making the flow highly heterogeneous and difficult to model accurately. Conventional discrete fracture network (DFN) models can represent this behavior, but they are often computationally demanding because they rely on mesh-based numerical methods. The main purpose of the study is therefore to combine DFN models with the analytic representation provided by the analytic element method (AEM), producing a mesh-free, three-dimensional model that yields continuous hydraulic head and velocity fields, and supports efficient particle tracking. Additional objectives include applying the developed model to various test cases and coupling it with an equivalent continuum model.

The developed model, implemented as the open-access Python package AnDFN, represents fractures as circular planar features in three-dimensional space and provides an analytic solution for groundwater flow within a fracture network. The formulation assumes an impermeable rock matrix, an incompressible fluid, and steady, irrotational flow, reducing the governing physics to Laplace's equation. Fracture boundaries, intersections, and specified hydraulic conditions are represented using Taylor series and asymptotic expansions, with the unknown coefficients solved iteratively. A key advantage of the approach is that, unlike mesh-based methods, it provides hydraulic head and velocity continuously throughout the fracture network, which is particularly valuable for accurate particle tracking and flow-path analysis. The method also provides a quantitative measure of numerical performance through the evaluation of boundary-condition residuals. These residuals quantify how closely the prescribed boundary conditions are satisfied and can be used to assess convergence.

By explicitly quantifying the boundary-condition residuals for a given DFN, the method provides a useful tool for the verification of numerical models. Models can be compared against a solution for which the degree of boundary-condition satisfaction is known, enabling the verification of models for complex fracture networks. The developed model was used to verify the commercial software FracMan® for three different DFNs consisting of 10, 100, and 1 000 fractures. All three cases show good agreement between the AnDFN model and FracMan®. The largest discrepancy can be attributed to the difference in the geometric representation of the fractures. In this study, the fractures in FracMan® are represented as 64-sided polygons, whereas in AnDFN they are approximated as perfect circles. This difference in geometry leads to deviations in the calculated intersection lengths.

Another contribution of the project is the integration of AnDFN with DarcyTools, a continuum groundwater flow model used for fractured media. This coupling enables detailed DFN-based simulation in selected subdomains while retaining an equivalent continuum model (ECM) elsewhere. The two models are linked iteratively by exchanging hydraulic heads and fluxes at their interface until convergence is achieved. A proof-of-concept

example demonstrates that a DFN embedded inside a DarcyTools domain can reproduce a consistent steady-state hydraulic gradient and allow particles to move continuously between the fractures and the continuum subdomains.

The report also presents practical applications, particularly tunnel inflow analysis and hydraulic upscaling. In the tunnel case, the model is used to simulate flow into a tunnel represented as a polygonal prism intersecting a stochastic fracture network. Across 50 realizations, the results show that tunnel inflow is spatially variable and strongly dependent on fracture connectivity, not only on fracture transmissivity. This suggests that DFN models can provide useful insights into inflow variability and potentially support grouting design and risk assessment. In the upscaling study, DFN realizations are used to estimate equivalent hydraulic conductivity tensors and porosity values for use in continuum models. The findings indicate that continuum models can reproduce some hydraulic behaviors and parts of travel-time distributions, but they may still miss important preferential flow paths and localized connectivity effects that could be captured by the statistical DFN representation.

In conclusion, the report demonstrates that AnDFN is a promising new tool for groundwater flow and transport modeling in fractured rock. Its main strengths are its mesh-free analytic formulation, continuous velocity field, and low boundary-condition residuals. At the same time, there are several limitations and future development needs, including improved solver efficiency for very large fracture networks, tighter source-code integration with DarcyTools, and enhanced methods for upscaling.

Keywords: Discrete Fracture Networks, Analytic Element Method, Equivalent Continuum Model, Verification

SAMMANFATTNING

Denna studie presenterar utvecklingen av en ny analytisk modell för att simulera grundvattenflöde och transport i sprickigt kristallint berg. Arbetet behandlar en stor utmaning inom hydrogeologi och bergteknik, nämligen att det mesta grundvattenflödet i kristallint berg sker genom sammanlänkade sprickor snarare än genom den omgivande bergmassan, vilket gör flödet starkt heterogent och svårt att modellera. Konventionella diskreta spricknätverksmodeller (DFN) kan representera detta beteende, men är ofta beräkningsmässigt krävande eftersom de bygger på beräkningsnätsbaserade numeriska metoder. Studiens huvudsakliga syfte är därför att kombinera DFN med den analytiska representation som används i analytiska elementmetoden (AEM), vilket resulterar i en beräkningsnätfri tredimensionell modell som ger kontinuerliga hydrauliska trycknivåer och hastighetsfält samt möjliggör effektiv partikelspårning. Ytterligare mål inkluderar att tillämpa den utvecklade modellen på olika testfall samt att koppla den till en ekvivalent kontinuummodell.

Den utvecklade modellen, implementerad som det öppna Python-paketet AnDFN, representerar en spricka som en cirkulär planär struktur i tredimensionellt rum och löser grundvattenflödet analytiskt inom spricknätverket. Formuleringen antar en tät bergmassa, en inkompressibel vätska samt stationärt, rotationsfritt flöde, vilket reducerar de styrande fysikaliska sambanden till Laplaceekvationen. Sprickgränser, korsningar och specificerade hydrauliska randvillkor representeras med hjälp av Taylorserier och asymptotiska serieutvecklingar, där de okända koefficienterna löses iterativt. En fördel med metoden är att den, till skillnad från beräkningsnätsbaserade metoder, ger hydrauliskt tryck och hastighet kontinuerligt i hela spricknätverket, vilket är särskilt värdefullt för noggrann partikelspårning och analys av flödesvägar. Metoden ger också ett kvantitativt mått på den numeriska prestandan genom utvärdering av residualer i randvillkoren. Dessa residualer kvantifierar hur väl de föreskrivna randvillkoren uppfylls och kan användas för att bedöma lösningens konvergens.

Genom att explicit kvantifiera randvillkorsresidualerna för ett givet DFN utgör metoden ett användbart verktyg för verifiering av numeriska modeller. Modeller kan jämföras med en referenslösning där graden av uppfyllelse av randvillkoren är känd, vilket möjliggör en utvärdering även av komplexa spricknätverk. Den utvecklade modellen användes för att verifiera den kommersiella programvaran FracMan® för tre olika DFN bestående av 10, 100 respektive 1 000 sprickor. Samtliga visade god överensstämmelse mellan AnDFN-modellen och FracMan®. Den största avvikelsen kan hänföras till skillnaden i geometrisk representation av sprickorna. I denna studie representeras sprickorna i FracMan® som 64-sidiga polygoner, medan de i AnDFN approximieras som perfekta cirklar. Denna skillnad i geometri leder till avvikelser i de beräknade skärningslängderna.

Projektet har även utvecklat en integration mellan AnDFN och DarcyTools, en kontinuerlig grundvattenmodell för sprickiga medier. Denna koppling möjliggör detaljerad DFN-baserad simulering i utvalda delområden samtidigt som en ekvivalent kontinuummodell (ECM) används i övriga delar. De två modellerna kopplas iterativt genom att utbyta hydrauliska trycknivåer och flöden vid deras gränssnitt tills konvergens uppnås. Ett demonstrationsfall visar att ett DFN inbäddat i en DarcyTools-domän kan återge en konsistent

stationär hydraulisk gradient och möjliggöra att partiklar rör sig kontinuerligt mellan sprickor och kontinuumsubdomänerna.

Rapporten presenterar även praktiska tillämpningar, särskilt inom tunnelinflödesanalys och hydraulisk uppskalning. I tunnelanalysen används modellen för att simulera flöde in i en tunnel representerad som ett polygonalt prisma som skär ett stokastiskt spricknätverk. Över 50 realiseringar visar resultaten att tunnelinflödet är rumsligt varierande och starkt beroende av sprickkonnektivitet, inte enbart av spricktransmissivitet. Detta tyder på att DFN-modeller kan ge värdefulla insikter om variationer i inflöde och potentiellt stödja injekteringsdesign och riskbedömning. I uppskalningsstudien används DFN-realiseringar för att uppskatta ekvivalenta hydrauliska konduktivitetstensorer och porositetvärden för användning i kontinuummodeller. Resultaten visar att kontinuummodeller kan återge vissa hydrauliska beteenden och delar av transporttidsfördelningen, men att de fortfarande kan missa viktiga preferentiella flödesvägar och lokala konnektivitetseffekter som kan fångas av en statistisk DFN-representation.

Sammanfattningsvis visar rapporten att AnDFN är ett lovande nytt verktyg för modellering av grundvattenflöde och transport i sprickigt berg. Dess huvudsakliga styrkor är den beräkningsnätfria analytiska formuleringen, det kontinuerliga hastighetsfältet och de låga randvillkorsresidualerna. Samtidigt finns det flera begränsningar och behov av vidareutveckling, inklusive förbättrad beräkningseffektivitet för mycket stora spricknätverk, tätare källkodsintegration med DarcyTools samt vidareutvecklade metoder för uppskalning.

Nyckelord: Diskreta spricknätverk, Analytic Element Method, Ekvivalent kontinuummodell, Verifiering

ABBREVIATIONS

List of acronyms

AEM	Analytic element method
DFN	Discrete fracture network
ECM	Equivalent continuum model
FEM	Finite element method
FVM	Finite volume method
REV	Representative elementary volume

List of variables

A	the cross-sectional area
e	the fracture aperture
g	the acceleration due to gravity
H	the effective thickness of the fracture plane
i	the imaginary unit
k	the hydraulic conductivity of a fracture
k_1	the major principal hydraulic conductivity
k_3	the minor principal hydraulic conductivity
k_{avg}	the average hydraulic conductivity derived from the tensor
k_i	the equivalent hydraulic conductivity in direction i
$k_{\text{measured},i}$	the measured hydraulic conductivity in direction i
$k_{\text{predicted},i}$	the predicted hydraulic conductivity in direction i
L	the characteristic length
n	the porosity
n_{fast}	the porosity based on first arrival time
n_{median}	the porosity based on median travel time
n_j	the normal vector of fracture j
N	the number of samples or directions
Q	the discharge
Q_{DFN}	the discharge of the DFN model
Q_{ECM}	the discharge of the ECM model
Q_i	the discharge in direction i
Q_p	the panel inflow
Q_t	the tunnel inflow
Q_x	the x component of the discharge vector
Q_y	the y component of the discharge vector
q_x	the x component of the specific discharge vector
q_y	the y component of the specific discharge vector
r	the fracture radius
$\tilde{r}$	the tunnel radius
t	the travel time

t_{fast}	the first arrival time
t_{median}	the median arrival time
T	the fracture transmissivity
u_j	the local real unit vector for fracture j
v_j	the local imaginary unit vector for fracture j
V	the volume of the control volume
x_i	the i -th component of a point in space
x_j	the center point of a fracture j
x_0	
x, y	the local fracture-plane coordinates
z	the complex variable
α_n	the n^{th} coefficient of the Taylor series
β_n	the n^{th} coefficient of the asymptotic expansion
$\Delta\phi$	the difference in hydraulic head
Δs	the step size in particle tracking
$\delta\phi$	the relative difference in hydraulic head
δQ	the relative difference in discharge
ζ	the distance between tunnel and groundwater surface
μ	the dynamic viscosity
ν_i	the velocity component in direction i
ϕ	the hydraulic head
ϕ_{DFN}	the hydraulic head for the DFN model
ϕ_{ECM}	the hydraulic head for the ECM model
Φ	the discharge potential
ξ	the tolerance in the particle tracking
Ψ	the stream function
ρ	the fluid density
Ω	the complex discharge potential

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1 INTRODUCTION

Modeling groundwater flow in fractured crystalline rocks is important for a wide range of rock engineering and hydrogeological applications, including rock tunneling (Piña et al., 2022), nuclear waste disposal (Follin & Hartley, 2014; Tsang et al., 2015), geothermal energy extraction (Fox et al., 2016), and contaminant transport assessment (Becker & Shapiro, 2003). In fractured crystalline rocks, the intact rock matrix usually exhibits low permeability; therefore, most measurable fluid flow occurs through interconnected fractures (National Research Council, 1996; Neuman, 2005). As a result, the hydraulic behavior of fractured rocks is strongly governed by the geometry, connectivity, and hydraulic properties of the fracture network (Maillot et al., 2016).

Because groundwater flow occurs predominantly within fractures, it is important to explicitly represent the fracture system when modeling groundwater flow in fractured crystalline rocks. In many geological settings, the contribution of the rock matrix to flow may be negligible compared with that of fractures, and the resulting flow field is therefore controlled by the fracture network, which determines preferential flow paths, hydraulic connectivity, and mass transport behavior across multiple spatial scales. Although continuum-based approaches, such as the equivalent continuum model (ECM), are computationally attractive, they often fail to capture the heterogeneous and anisotropic nature of fractured rock systems. For example, Niemi et al. (2000) demonstrated that continuum models frequently cannot reproduce the correct flow behavior in fractured crystalline rocks, particularly when flow localization and fracture connectivity dominate the hydraulic response. Consequently, discrete fracture network (DFN) models have been widely used to represent groundwater flow in fractured crystalline rocks.

Despite their conceptual advantages, three-dimensional DFN models remain challenging to develop and apply in practice. The complexity arises from the need to explicitly represent large numbers of fractures with varying sizes, orientations, apertures, and intersections in a three-dimensional domain (de Dreuzy et al., 2012; Hyman et al., 2015). Numerical simulation of flow in such systems often requires computationally intensive meshing procedures and the solution of large-scale algebraic equation systems, especially when fracture density and geometric complexity increase. Traditional numerical methods, such as finite element (FEM) and finite volume (FVM) methods, may suffer from high computational costs, mesh-generation difficulties, and convergence issues associated with complex fracture intersections and multi-scale geometries. These challenges become even more pronounced for stochastic DFN realizations, uncertainty quantification, inverse modeling, and full site-scale engineering applications where repeated simulations are required. In particular, particle tracking in three-dimensional DFN models remains computationally demanding and numerically challenging. Accurate particle tracking requires detailed resolution of the velocity field within individual fractures and across fracture intersections (Makedonska et al., 2015). In mesh-based numerical methods, particle trajectories are typically calculated from interpolated nodal or element-based velocities, which may introduce numerical dispersion, interpolation errors, and sensitivity to mesh quality and discretization density (Makedonska et al., 2015).

These limitations motivate the development of alternative modeling approaches that provide continuous velocity fields and efficient particle-tracking formulations for groundwater flow and mass transport in fractured rock systems. To address these limitations, there is increasing interest in developing efficient and physically robust analytic approaches for groundwater flow in three-dimensional DFNs. Among such approaches, the analytic element method (AEM) offers several attractive features for modeling groundwater flow in fractured rock systems (Strack & Toller, 2024). Specifically, AEM represents the hydraulic field through analytic solutions that satisfy the governing equations exactly within the model domain, thereby avoiding domain-wide discretization and reducing numerical complexity (Strack, 1989). This mesh-free characteristic makes AEM particularly appealing for problems involving complex fracture geometries and large spatial domains. Furthermore, AEM provides a quantitative measure of numerical performance through the evaluation of boundary-condition residuals, making it suitable as a verification tool for numerical models.

1.1 Aim and Objectives

In this study, we aim to develop a novel three-dimensional DFN model for flow and transport simulations based on analytic functions using the AEM formulation presented by Strack and Toller (2024). The developed analytic model seeks to combine explicit DFN representations with the computational efficiency and mathematical advantages of analytic formulations. In particular, the study aims to establish a modeling approach in which the velocity field is described continuously throughout the fracture network, thereby improving the representation of groundwater flow and particle tracking in fractured rock systems.

To achieve this aim, the following specific objectives are defined:

1. The primary objective is to develop a novel three-dimensional DFN model for groundwater flow simulations based on analytic functions.
2. A second objective is to incorporate a particle-tracking formulation within the developed analytic model. By utilizing the continuous velocity field obtained from the analytic solution, the model seeks to improve the simulation of groundwater flow paths and travel times in fractured rocks.
3. A third objective is to establish an integration strategy between the developed DFN model and the ECM groundwater flow model DarcyTools. This integration is achieved through an iterative exchange of boundary conditions using source and sink terms, thereby enabling the representation of detailed fracture-scale flow processes within larger hydrogeological systems.
4. A fourth objective is to evaluate the use of the developed analytic DFN model as a verification tool for existing DFN simulation approaches. The verification focuses on hydraulic head distributions and breakthrough curves derived from particle-tracking simulations.
5. The final objective is to demonstrate the applicability and practical potential of the

developed analytic DFN model through representative flow and transport applications. These examples are intended to illustrate the capability of the model to simulate groundwater flow and transport processes in fractured rock systems, as well as its suitability for engineering and environmental applications involving complex three-dimensional fracture networks.

1.2 Modeling concepts

Fractured rock formations differ fundamentally from intergranular geological formations, such as sands and gravels, because flow in fractured rock is primarily concentrated within fractures, while the intact rock matrix often contributes relatively little to the overall flow (National Research Council, 1996; Neuman, 2005; Tsang et al., 2015). The resulting flow field is therefore highly heterogeneous, anisotropic, and strongly influenced by the geometry and connectivity of the fracture system. Individual fractures may act as preferential flow pathways, while fracture intersections form hydraulic connections that govern the movement of water through the network (Cacas et al., 1990; Maillot et al., 2016). Consequently, groundwater flow in fractured rocks is controlled not only by hydraulic properties but also by the spatial distribution of fractures (Bear, 1972; Cacas et al., 1990).

Several conceptual approaches have been developed to represent groundwater flow in fractured rock environments. These approaches differ primarily in the way fractures are represented and the level of detail used to describe the flow system. A comprehensive review and comparison of these different conceptual approaches can be found in the literature (Berre et al., 2018; Selroos et al., 2002). In this study, we focus only on the DFN and ECM approaches.

Discrete fracture network

The DFN approach explicitly represents individual fractures and their intersections (Long et al., 1982). Each fracture is characterized by geometric and hydraulic properties, such as location, orientation, size, aperture, transmissivity, and storativity. Groundwater flow is simulated directly within the fracture network, allowing preferential flow paths and connectivity effects to emerge naturally from the fracture geometry (Dershowitz & Herda, 1992; Follin & Hartley, 2014; Hyman et al., 2015).

In DFN models, individual fractures are commonly represented as two-dimensional planar or undulating surface features embedded within a three-dimensional rock domain. Flow within a fracture is often described using a hydraulic law, such as the cubic law (Snow, 1965), which relates fracture transmissivity to fracture aperture under the assumption of laminar flow between parallel plates. In fracture networks, continuity of hydraulic head and conservation of mass must be satisfied at fracture intersections. The resulting DFN flow problem therefore consists of a set of coupled flow equations defined on individual fracture planes and linked through intersection conditions.

Most existing DFN models employ numerical discretization techniques such as finite element and finite volume methods (de Dreuzy et al., 2012; Hyman et al., 2015). These ap-

proaches approximate the governing equations on computational meshes and solve large systems of algebraic equations. While highly versatile, numerical methods often require extensive mesh generation, substantial computational resources, and repeated interpolation of velocity fields for particle tracking, especially for large three-dimensional networks (Makedonska et al., 2015).

An alternative strategy is to formulate the DFN flow problem using analytic or semi-analytic solutions. In such approaches, hydraulic head and velocity fields are represented by analytic functions that satisfy the governing equations directly (Long et al., 1985; Strack & Toller, 2024). Analytic formulations can reduce computational requirements, eliminate mesh-related errors, and provide continuous velocity fields throughout the fracture network. These characteristics are particularly advantageous for particle tracking and transport simulations, where accurate velocity information is required along flow paths. The analytic method developed in this study follows this philosophy. By combining explicit DFN representations with analytic flow solutions, the model seeks to retain the discrete nature of fracture networks while improving computational efficiency and facilitating particle tracking in complex three-dimensional fractured rock systems.

Equivalent continuum model

The ECM represents fractured rock as a continuum with effective hydraulic properties that average the influence of fractures over a representative elementary volume (REV) (Bear, 1972; Zhang & Sanderson, 2002). Instead of explicitly representing each fracture, the hydraulic effects of the fracture network are homogenized into equivalent parameters, such as hydraulic conductivity, transmissivity, and storativity, which are assigned to continuum grid cells (Oda, 1985). Groundwater flow is then governed by Darcy's law and the corresponding groundwater flow equation in a manner similar to conventional porous media (Bear, 1972).

This approach is computationally efficient and is widely used in groundwater studies where the domain dimensions are much larger than the characteristic fracture spacing (Adler et al., 2012; Zhang & Sanderson, 2002). The ECM is particularly suitable when the fracture network is sufficiently dense and well-connected that flow behavior can be adequately described by averaged hydraulic properties and a REV can be defined. Under these conditions, the detailed geometry of individual fractures has a limited influence on the studied flow field, and continuum approximations can provide accurate predictions of hydraulic heads and groundwater fluxes (Hadgu et al., 2017).

A major advantage of the ECM is its ability to simulate groundwater flow over large spatial and temporal scales while maintaining manageable computational requirements (Follin & Thunvik, 1994; Zhang & Sanderson, 2002). Consequently, continuum models are commonly employed for regional hydrogeological assessments, groundwater resource evaluations, and site investigations. However, because fractures are not represented explicitly, ECM approaches may not capture detailed flow channeling, preferential pathways, fracture connectivity effects, or localized transport phenomena that often govern groundwater movement in fractured crystalline rocks (Tsang & Neretnieks, 1998).

Despite these limitations, continuum models play an important role in fractured rock hydrogeology because they provide the hydraulic boundary conditions that drive local fracture-scale flow processes. In practical applications, continuum models are often coupled with more detailed DFN models (referred to as a hybrid model), allowing regional groundwater flow patterns to be represented efficiently while preserving local-scale fracture heterogeneity where required. In this study, we further develop such a hybrid model by coupling an AEM-based DFN with the continuum model DarcyTools. Hence, in the context of groundwater flow modeling in fractured rock, DarcyTools can provide regional hydraulic heads, flow directions, and boundary conditions that may subsequently be used in an AEM-based DFN model for detailed analyses of local flow and transport processes. This multiscale modeling strategy combines the computational efficiency of continuum approaches with explicit fracture representations.

DarcyTools

DarcyTools was specifically developed for simulating flow in fractured media using a FVM solver (Svensson & Ferry, 2010). It solves the groundwater flow equation on a continuum grid using Darcy's law and can represent heterogeneous and anisotropic hydraulic conductivity fields derived from geological and hydrogeological data. In particular, DarcyTools incorporates several unique features for groundwater modeling in fractured crystalline bedrock, including the generation of DFNs that are upscaled to an ECM.

In this study, a Python wrapper for DarcyTools has been developed. This Python wrapper can

- store and save compact input files (so-called *cif*-files),
- write the *cif*-files,
- run DarcyTools executables,
- import results and perform particle tracking, and
- plot the results or other parameters in three-dimensional space.

Because the wrapper is written in Python, it works seamlessly with other Python packages, including the Python package developed in this study for DFN flow and transport modeling.

1.3 Data availability statement

The DFN model developed in this study is available as an open-access Python package (Toller, 2026), and the reader is referred to <https://eriktoller.github.io/andfn/> for more information.

2 METHODOLOGY

The theory behind the developed three-dimensional DFN model using the AEM is presented in detail in Strack and Toller (2024); herein, a brief summary is included.

2.1 Analytic element method

The analytic element method (AEM) is an analytic approach for simulating groundwater flow. The AEM was initially developed in the late 1970s for regional groundwater flow, but has since been extended to transient, three-dimensional flow and linear elasticity (Bakker, 2004; Jankovic et al., 2006; Strack, 1989; Strack & Toller, 2022). A key feature of AEM is that it is mesh-free: the solution is constructed as a superposition of analytic functions that satisfy the governing equations exactly (up to truncation of series expansions), rather than through spatial discretization.

In contrast to mesh-based methods such as FEM, where the governing equations are formulated in weak form, and the numerical accuracy depends on spatial discretization, AEM constructs the solution as the sum of the contributions from individual analytic elements, yielding a mesh-free continuous field that can be evaluated directly throughout the domain. This makes it possible to represent features at widely different spatial scales within one formulation by adding analytic elements, without the mesh refinement requirements typical of FEM. However, AEM is still approximate overall because boundary and interface conditions are usually satisfied numerically rather than exactly, so its accuracy depends on element selection, series order, and collocation/integration strategy rather than on mesh resolution.

Moreover, there is no theoretical limit to the number of elements that can be included in an AEM model. Because the governing formulation is analytic, numerical precision is constrained primarily by machine precision and series truncation rather than by grid resolution. This capability is demonstrated by Barnes and Janković (1999) and Strack and Toller (2022), where models containing tens of thousands of elements maintain accuracy in both the near and far field.

An AEM model is constructed by combining different analytic elements, each representing a specific feature of the flow domain and assigned an associated boundary condition. The AEM preserves its analytic characteristics even for complex geometries by relying on powerful mathematical techniques. The boundary condition is mapped, together with the element, from the physical z -plane to the transformed χ -plane, where the element is typically represented as the unit circle and expressed as a power series, i.e., a Taylor series or an asymptotic expansion, whose unknown coefficients can, in many cases, be solved using the Cauchy integral.

2.2 Analytic model for flow in a DFN

The basic theory behind solving the analytic model for flow in a DFN is presented in Strack and Toller (2024). The model is derived under the following simplifying assumptions:

- the rock matrix is impermeable,
- the fluid is incompressible, and
- the flow is both divergence-free and irrotational.

These assumptions imply steady-state groundwater flow, meaning that there is no temporal change in hydraulic head and no change in aquifer storage over time. Together, these conditions reduce the governing equations to Laplace's equation:

$$\nabla^2 \Phi = 0 \quad (1)$$

where Φ is the discharge potential. It should be noted that the assumptions adopted in this study are not universally valid for all groundwater systems, but are instead specific simplifications required to obtain an analytic model. In more general cases, such as transient flow, compressible fluids, or systems with storage, the governing equations become more complex, and numerical methods beyond the one presented in this study are required.

The DFN is defined in the three-dimensional Cartesian space, x_j , where $j = 1, 2, 3$. Each fracture is represented by an infinite plane of finite thickness. The modeled fracture domain is bounded by a circle and is defined by a unit normal n_j and two orthogonal unit vectors parallel to the plane, v_j and u_j . The vectors v_j and u_j are orthogonal to each other and together form the local (x, y) Cartesian space. The representation of a fracture in three- and two-dimensional space is illustrated in Figure 1. In the two-dimensional space, the local x - and y -axes that make up the complex plane are the same unit vectors that define the fracture plane in three-dimensional space. In the two-dimensional fracture plane, a complex variable is introduced

$$z = x + iy \quad (2)$$

where i is the imaginary unit defined as $i = \sqrt{-1}$.

A point in the three-dimensional space is mapped onto the fracture centered at x_j using the two dot products

$$x = \begin{pmatrix} x_j - x_j \\ 0 \end{pmatrix} \cdot u_j \quad (3)$$

and

$$y = \begin{pmatrix} x_j - x_j \\ 0 \end{pmatrix} \cdot v_j. \quad (4)$$

The three-dimensional coordinates corresponding to a point in the two-dimensional fracture plane are given by the projection:

$$x_j = xu_j + yv_j + x_j. \quad (5)$$

The flow within each fracture can be described with a complex potential

$$\Omega = \Phi + i\Psi \quad (6)$$

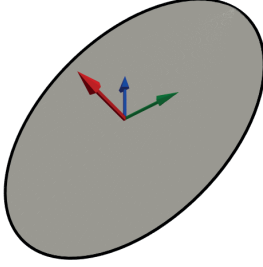
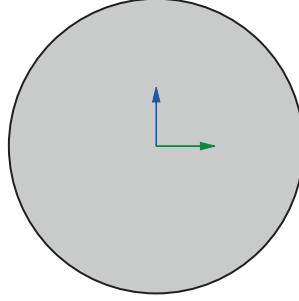
a**b**

Figure 1. A fracture in three-dimensional space with its normal (red) and two unit vectors (green and blue) (a), and its corresponding representation in two-dimensional space (b).

where Ψ is the stream function. The discharge potential is calculated using

$$\Phi = kH\phi \quad (7)$$

where k is the hydraulic conductivity, H is the thickness of the fracture, and ϕ is the hydraulic head. In the AEM formulation, H denotes the effective thickness of the fracture plane; in the present groundwater-flow context, this corresponds physically to the fracture aperture used in the flow calculations. The components of the specific discharge vector can be expressed using Darcy's law as:

$$q_x = -k \frac{\partial \phi}{\partial x} \quad (8)$$

and

$$q_y = -k \frac{\partial \phi}{\partial y}, \quad (9)$$

or in terms of the discharge vector as:

$$Q_x = q_x H = -\frac{\partial \Phi}{\partial x} \quad (10)$$

and

$$Q_y = q_y H = -\frac{\partial \Phi}{\partial y}. \quad (11)$$

The analytic elements used to represent the various boundary conditions are either the Taylor series, e.g., the fracture boundary,

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \chi(z)^n \quad (12)$$

or the asymptotic expansion, e.g., the intersection or constant-head boundary,

$$g(z) = \sum_{n=0}^{\infty} \beta_n \chi(z)^{-n}. \quad (13)$$

The unknown coefficients of these power series, α_n and β_n , are solved for, and the number of coefficients required is determined by the accuracy requirements and the complexity of the network.

Although it is not necessary to bound the fracture domain, as infinity can be, and often is, included in AEM models, the model developed in this study does so. Each fracture is bounded by a circle centered at the local origin. While it is, in principle, possible to confine the extent of a fracture to any geometric shape, the present study is limited to circular boundaries.

On the boundary of a fracture, a constant streamline condition, i.e., a no-flow boundary, is imposed using a Taylor expansion. Imposing a constant value of the stream function ensures that no flow crosses the boundary; i.e., the normal component of the specific discharge vanishes. Flow is therefore confined to be tangential to the boundary.

When two fractures intersect, they share a common intersection line. This intersection line is represented as an analytic element whose purpose is to transfer flow between the two fractures and to enforce continuity of hydraulic head across the intersection. This condition does not imply that the head is constant along the intersection. Rather, the head varies spatially along the line but remains continuous between the two fractures. The enforcement of continuity is achieved through the analytic element formulation by satisfying the condition at a set of control points along the intersection. Similarly, the mathematical formulation of the element ensures that the flow is continuous between the two fractures.

The flow solution is obtained through an iterative procedure. The solver is divided into two parts: one that determines the discharges at different discharging elements, e.g., intersection elements and head-specified elements, and another that solves for the coefficients of the various series expansions. The iterative process is illustrated in Figure 2.

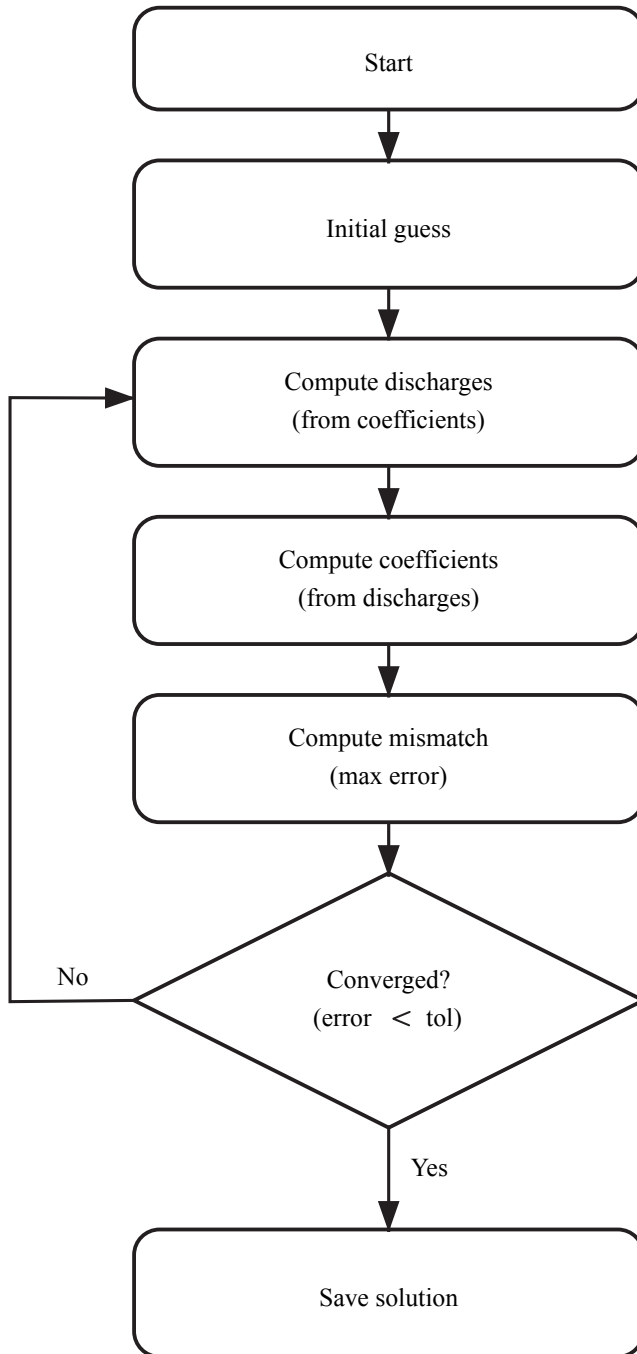


Figure 2. Illustration of the iterative solver process.

Assessment of solution accuracy

The numerical accuracy of the AEM model is assessed by evaluating the residuals in the boundary conditions of the analytic elements. These residuals arise from the truncation of the power series representations used in the solution and provide a measure of how closely the prescribed boundary conditions are satisfied.

Because there generally does not exist an exact solution for a DFN, the exact error throughout the domain cannot be determined directly. Consequently, the present analysis does not provide a rigorous upper bound on the solution error. Instead, the reported error represents the degree to which the boundary conditions are satisfied by the approximate solution. Small boundary-condition residuals indicate that the approximate solution closely satisfies the governing boundary conditions and therefore indicate numerical convergence and solution accuracy.

It is important to note that this error measure only reflects numerical performance; errors arising from the conceptualization of the groundwater system, parameter uncertainty, or boundary-condition uncertainty may still be present.

Implementation

The developed analytic DFN model has been implemented and published as a Python package called AnDFN¹. The model is organized around a central DFN class, into which individual fracture instances are incorporated. Each fracture instance maintains a collection of associated analytic elements, including the bounding circle that forms the fracture edge boundary, constant-head line elements, wells, impermeable boundaries, and related features. These fractures can either be generated or imported from a CSV file.

It is also possible to add structures to the DFN. These structures are prism-based representations of cylindrical features and may either be constant-head elements (e.g., wells or tunnels) or impermeable elements (e.g., concrete structures). The model automatically computes intersections between fractures and structures and assigns the appropriate elements to each affected fracture.

Fractures in the DFN can be generated, imported, or defined directly by the user. The built-in DFN generator is relatively simple and currently supports only a uniform random distribution. However, many advanced DFN generators already exist, and DFNs generated by such tools can be imported using a CSV file. In this study, FracMan[®] was used to generate the DFNs, which were subsequently imported into AnDFN.

Once the fracture network is loaded and has undergone an intersection check, the flow solution can be computed. The AnDFN solver is coded in a dedicated module that generates compiled code for fast execution using the Python package Numba (Lam et al., 2015). The solver operates fully automatically, but the user may adjust several parameters, such as the error tolerance, maximum number of iterations, and other settings that control the

¹<https://eriktoller.github.io/andfn/>

accuracy and computational speed. When the solver has finished, it returns metrics describing the solution, and the computed discharges and coefficients are stored for each element.

The final functionality of the developed code is plotting and particle tracking. Plotting is performed by creating a mesh within each fracture, on which the complex potential is computed at every node. It should be noted that this mesh is used solely for visualization purposes. Once the complex potential has been evaluated, the hydraulic head or the flow net can be contoured across the mesh. The plotting routine is implemented using the Python package PyVista (Sullivan & Kaszynski, 2019).

An important feature of the model is that it is parallelized and can utilize all available threads. Nearly all components of the solver and plotter are parallelized, with the exception of certain high-level data management tasks. The current speed-up achieved when solving a fracture network is shown in Figure 3. The DFN used in this performance test consisted of 2 781 fractures and 7 830 elements.

The results indicate that there is still considerable room for improvement in the parallelization and that a substantial portion of the total computational time is currently spent in high-level data management routines and memory transfer. It is also worth noting that the plotter does not show any noticeable speed-up, likely due to the overhead overshadowing the computational part.

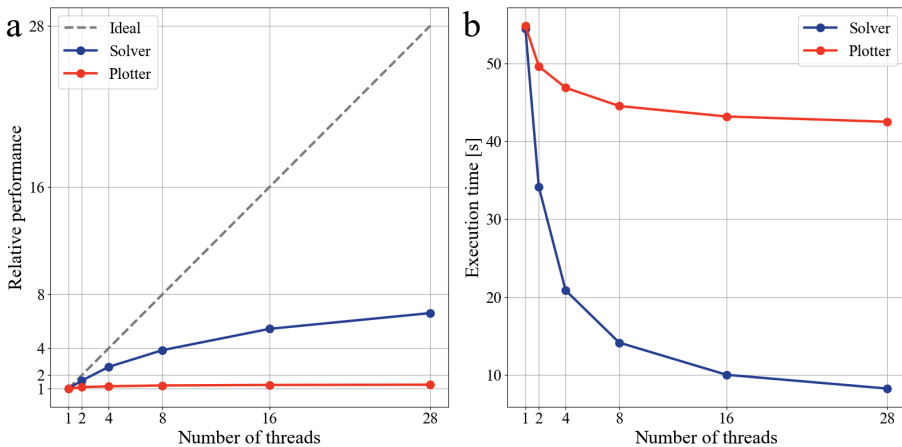


Figure 3. Parallel performance of the solver and plotting routines as a function of the number of threads. Relative performance improvement compared to single-thread execution (a) and the corresponding execution times (b).

2.3 Integration between DFN and ECM

Combining a DFN with an ECM implies that the groundwater flow transitions from a three-dimensional grid-based representation to a two-dimensional fracture-plane representation. When moving from an ECM grid model to a DFN, more precisely to a single fracture extracted from the three-dimensional domain, the boundary conditions must be treated with particular care. The boundary conditions at the interface between the ECM and the fractures are:

- continuity of hydraulic head
- continuity of discharge

These conditions can be expressed as:

$$\phi_{\text{ECM}} = \phi_{\text{DFN}} \quad (14)$$

and

$$Q_{\text{ECM}} = Q_{\text{DFN}} \quad (15)$$

where ϕ_{ECM} and ϕ_{DFN} are the hydraulic heads of the ECM and DFN at the interface line, and Q_{ECM} and Q_{DFN} are the total discharges of the ECM and DFN at the interface line. These conditions represent the standard coupling requirements for flow across interfaces commonly applied in groundwater flow models. The challenge in the present approach is not the form of the boundary conditions themselves, but their implementation for accurately capturing flow exchange between the ECM and the DFN. By enforcing a fixed discharge with a free hydraulic head on one side and a fixed head with a free flow on the other, one condition can be imposed as a boundary condition, while the other serves as a convergence criterion.

In this procedure, the DFN and ECM are not directly linked in the source code; instead, they are connected through a post-processing workflow. This means that the two models must be run to completion multiple times, i.e., to steady-state conditions, in sequence to reach a converged solution. The iterative process is illustrated in Figure 4. Convergence is assessed by comparing the discharges and hydraulic heads at the interface between the two models. This is done using relative errors, $\delta\phi$ and δQ , defined as

$$\delta\phi = \left| \frac{\phi_{\text{ECM}} - \phi_{\text{DFN}}}{\phi_{\text{DFN}}} \right| \quad (16)$$

and

$$\delta Q = \left| \frac{Q_{\text{ECM}} - Q_{\text{DFN}}}{Q_{\text{DFN}}} \right|. \quad (17)$$

A notable difference between this ECM–DFN coupled model and other existing approaches (e.g., FracMan®) is that only one of the components is mesh-based, while the other, AnDFN, is mesh-free. This naturally introduces a specific challenge at the interface, where information defined on a mesh must be transferred to a mesh-free analytic element representation.

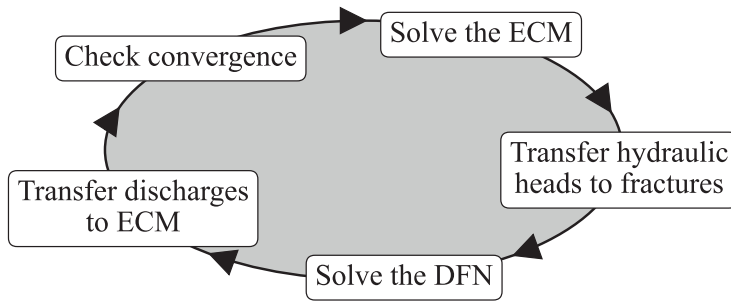


Figure 4. The conceptual model of the iteration process between the DFN and the ECM.

Linking the models

To link the two models, a boundary condition must be defined in each of them. In the AnDFN model, this is implemented using a constant-head line element, where the specified head is taken directly from a point, or more precisely a DarcyTools spot, located at the center of the constant-head line element along the DFN–ECM interface in DarcyTools. In DarcyTools, the same spot is assigned a prescribed flux corresponding to the total discharge across the constant-head line element. An illustration of this setup is shown in Figure 5. Because AnDFN currently does not support fixed-flow line sources, the constant-head condition is applied in AnDFN, while the constant-flux condition is applied in DarcyTools. The point is used in DarcyTools to avoid overlapping boundary conditions in the grid.

The method is not restricted to a single spot per fracture–continuum intersection. Instead, the interface line can be divided into multiple segments, each associated with its own spot, allowing a more detailed and flexible representation of the exchange between the ECM and the DFN. In this study, however, a single spot was used. The underlying concept is similar to the coupling strategy described by Joyce et al. (2010).

It is important to note that replacing a portion of an isotropic ECM with an explicit DFN representation may alter the local hydraulic behavior. While the ECM represents the rock mass as a homogeneous and isotropic continuum, the DFN explicitly accounts for individual fractures and their connectivity. Consequently, the regional-scale flow field may shift in response to the more detailed representation of flow within the replaced region.

Limitations of the approach

The approach is limited by the capabilities of the two solvers. It is important to note that importing and exporting data between the two models is time-consuming, which reduces the overall computational feasibility of the method. However, as a proof-of-concept, it

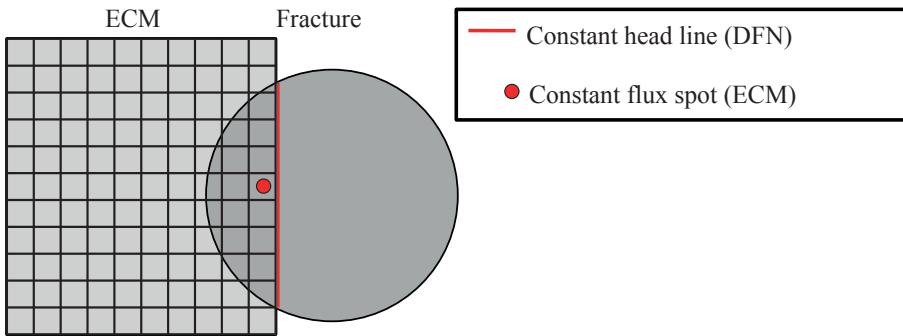


Figure 5. An illustration of the boundary condition representation in the fracture and the ECM mesh.

clearly demonstrates that the two models can, in fact, be combined. For future implementations, it would be more efficient to link the two models directly at the source code level.

One limitation of using spots in DarcyTools is that overlapping spots do not automatically sum their discharges; instead, the last spot added determines the applied discharge. For sparse fracture networks, this is unlikely to be a major issue, but for dense networks, it may lead to inaccuracies. In addition, the number of spots grows with the number of fractures intersecting the ECM, which requires DarcyTools to impose a large number of boundary conditions and to export hydraulic head values at many locations.

There is also a conceptual limitation associated with the way the ECM and DFN are coupled. The contact area between a fracture and the surrounding continuum is small relative to the volume represented by the continuum. This creates points with a high flow rate in a low conductivity matrix, which is problematic. Moreover, the capture zone of an individual fracture is limited. As a result, it is important that the ECM is upscaled using the same DFN that is embedded in the model. Otherwise, the preferential flow paths will not match between the ECM and the DFN, leading to inconsistencies in the representation of groundwater flow.

Particle tracking

The particle-tracking algorithm is a post-processing step that uses the steady-state flow solution to compute particle trajectories. In both models, a n^{th} -order Runge-Kutta method is applied using Python. The theoretical background for particle tracking in both two- and three-dimensional flow fields is presented in Strack (1989).

Particle tracking in the ECM is conducted in two stages. First, Euler's method is used to compute an initial estimate of the next particle location, providing a first-order prediction of the trajectory. This predicted position is then refined using the higher-order Runge-Kutta scheme, which evaluates the velocity field at intermediate points to obtain a more

accurate final position for each tracking step. The initial step is calculated as:

$$x_i = x_i + \frac{\nu_i \begin{pmatrix} x_j \\ 1 \end{pmatrix}}{\left| \nu \begin{pmatrix} x_j \\ 1 \end{pmatrix} \right|} \Delta s \quad i = 1, 2, 3 \quad \text{and} \quad j = 1, 2, 3 \quad (18)$$

where x_i is the starting point component, ν_i is the velocity vector component, and Δs is the step size. Subsequent estimates of the particle location are calculated using:

$$x_i = x_i + \frac{\frac{1}{2} \left[\nu_i \begin{pmatrix} x_j \\ 1 \end{pmatrix} + \nu_i \begin{pmatrix} x_j \\ n-1 \end{pmatrix} \right]}{\nu^*} \Delta s \quad (19)$$

where

$$\nu^* = \frac{1}{2} \sqrt{\sum_{i=1}^3 \left[\nu_i \begin{pmatrix} x_j \\ 1 \end{pmatrix} + \nu_i \begin{pmatrix} x_j \\ n-1 \end{pmatrix} \right]^2}. \quad (20)$$

The iterative procedure is repeated until

$$\left| x_i - x_{i-1} \right| \leq \xi \quad (21)$$

where ξ is a given tolerance. In this work, a tolerance of 10^{-3} m is used.

The particles can be transferred between the DFN and the ECM by registering their exit location in three-dimensional space and determining the corresponding cell or fracture. During this process, a small positional adjustment may be necessary to ensure that the point falls within a valid fracture surface or ECM cell.

3 VERIFICATION

This chapter demonstrates the use of AnDFN as a verification tool for groundwater flow models in fractured rock. The model is first compared against an exact analytical solution and subsequently used as a high-accuracy reference model to assess the accuracy and consistency of other DFN modeling approaches.

3.1 Four-fracture model

The developed model is compared with an exact solution for flow in a small fracture network. The exact solution is adopted from Makedonska et al. (2015) and presented in Figure 6. The results presented herein are expressed in terms of transmissivity and hydraulic head rather than permeability and pressure, as in Makedonska et al. (2015). The solution considers one-dimensional flow from an inflow location with a fixed hydraulic head, ϕ_{in} , and two outflow locations, both with a fixed hydraulic head ϕ_{out} . The parameters of the problem are given in Table 1.

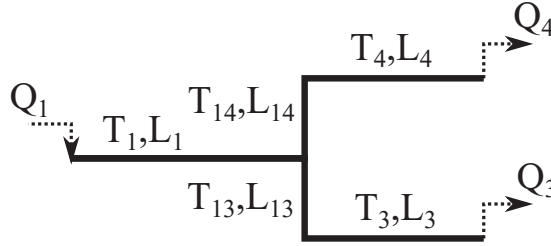


Figure 6. Schematic sketch of the four-fracture model, where the subscript numbers represent the fracture numbers.

Because the model represents fractures using circular boundaries, it does not exactly reproduce the geometry of the exact solution. Constant-head lines are used for the hydraulic head boundaries, and they span the entire width of the fracture. An illustration of the model with a fracture radius of 1 m is shown in Figure 7. A total of 200 coefficients were used for each element.

The results of the comparison are presented in Table 2, where the absolute difference in time, Δt , is defined as:

$$\Delta t = |t_{\text{exact}} - t_{\text{model}}| \quad (22)$$

where t_{exact} is the exact solution and t_{model} is the model result. For the distribution of the discharge, i.e., how much of the flow enters fractures 3 and 4, ΔQ is defined as:

$$\Delta Q = \left| \frac{Q_{3,\text{exact}}}{Q_{4,\text{exact}}} - \frac{Q_{3,\text{model}}}{Q_{4,\text{model}}} \right|, \quad (23)$$

Table 1. Parameter values given by Makedonska et al. (2015).

Property		Value
hydraulic head	ϕ_{in}	$= 200 \cdot 10^6 / (\rho g) \text{ m}$
	ϕ_{out}	$= 100 \cdot 10^6 / (\rho g) \text{ m}$
fluid path length	L_1	$= 0.9 \text{ m}$
	L_{13}	$= 0.2 \text{ m}$
	L_3	$= 0.9 \text{ m}$
	L_{14}	$= 0.2 \text{ m}$
	L_4	$= 0.9 \text{ m}$
transmissivity	T_1	$= 2.5 \cdot 10^{-13} \rho g e / \mu \text{ m}$
	T_{13}	$= 2.5 \cdot 10^{-13} \rho g e / \mu \text{ m}$
	T_3	$= 2.5 \cdot 10^{-13} \rho g e / \mu \text{ m (case 1)}$
	T_3	$= 2.5 \cdot 10^{-12} \rho g e / \mu \text{ m (case 2)}$
	T_{14}	$= 2.5 \cdot 10^{-13} \rho g e / \mu \text{ m}$
	T_4	$= 2.5 \cdot 10^{-13} \rho g e / \mu \text{ m}$
density	ρ	$= 1000 \text{ kg/m}^3$
standard acceleration of gravity	g	$= 9.81 \text{ m/s}^2$
fracture aperture	e	$= 0.01 \text{ m}$
viscosity	μ	$= 1.002 \text{ Pa}\cdot\text{s}$

where $Q_{3,\text{exact}}$ and $Q_{4,\text{exact}}$ are the flows for the exact solution, and $Q_{3,\text{model}}$ and $Q_{4,\text{model}}$ are the flows for the model. The relative difference in time, δt , is defined as:

$$\delta t = \left| \frac{t_{\text{exact}} - t_{\text{model}}}{t_{\text{exact}}} \right| \quad (24)$$

and the relative difference in the flow distribution, δQ , is defined as:

$$\delta Q = \left| \left(\frac{Q_{3,\text{exact}}}{Q_{4,\text{exact}}} - \frac{Q_{3,\text{model}}}{Q_{4,\text{model}}} \right) \left(\frac{Q_{3,\text{exact}}}{Q_{4,\text{exact}}} \right)^{-1} \right|. \quad (25)$$

The results show that the exact solution and the current model are in close agreement in both cases. The agreement is slightly weaker in the second case, which appears to be more sensitive to geometric representations. The maximum boundary-condition residual for the AnDFN solution is $2.880 \cdot 10^{-11} \%$, indicating that the observed differences are most likely a result of differences in the geometric representation rather than numerical errors in the model.

3.2 Random fracture networks

Because the AnDFN model provides a quantitative measure of how closely the governing boundary conditions are satisfied, it can serve as a verification tool for mesh-based numerical approaches. Its verification capability is demonstrated by using it as a reference

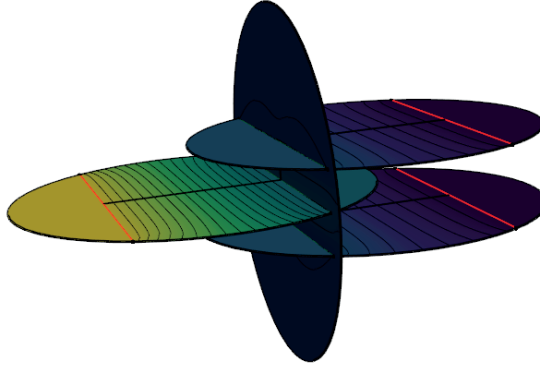


Figure 7. The implementation of the four-fracture model as circular fractures; the red lines represent the constant-head boundaries, the thin black lines are equipotential lines, and the thick black line(s) along the fracture centers are the tracked streamlines.

Table 2. Results of the comparison between the developed model and the exact solution; the variables Δt , δt , ΔQ , and δQ are defined in (22), (24), (23), and (25), respectively.

Test case	Travel time		Mass fraction	
	Δt	δt	ΔQ	δQ
Case 1, leg 3 & 4	$7.9 \cdot 10^0$	$4.4 \cdot 10^{-4}$	$2.5 \cdot 10^{-12}$	$5.0 \cdot 10^{-12}$
Case 2, leg 3	$3.1 \cdot 10^1$	$3.1 \cdot 10^{-3}$	$8.4 \cdot 10^{-3}$	$4.0 \cdot 10^{-2}$
Case 2, leg 4	$4.2 \cdot 10^2$	$1.5 \cdot 10^{-2}$	$8.4 \cdot 10^{-3}$	$1.1 \cdot 10^{-2}$

to verify the commercial software FracMan[®]. All DFNs used in this study are generated using FracMan[®], with the parameters presented in Table 3. The transmissivity of the fractures is defined as

$$T = 6.7 \cdot 10^{-9} r^{1.4} \quad (26)$$

where r is the fracture radius. The aperture, e , used in the particle-tracking calculations is estimated using the empirical relationship proposed by Dershowitz et al. (2003):

$$e = 0.46\sqrt{T}. \quad (27)$$

The minimum and maximum radii were set to 10 and 40 m, respectively, for the verification test cases. To achieve the desired number of fractures, each fracture set was initially generated with a limited count and then filtered for connected fractures. If the number of fractures did not match the target case number, the fracture sets were regenerated on top of the existing DFN until the desired number of fractures was obtained.

Table 3. Statistical parameters used for generating the DFNs (from Follin et al. (2007), Table 11-20: FFM01 > -200 m).

Fracture set name	Orientation set pole: (trend, plunge), Fisher conc.	Size model, power-law (r_0, k_r) (m, -)	Intensity P_{32} , open (m^2/m^3)
NS	(292, 1), 17.8	(0.038, 2.50)	0.073
NE	(326, 2), 14.3	(0.038, 2.70)	0.319
NW	(60, 6), 12.9	(0.038, 3.10)	0.107
EW	(15, 2), 14.0	(0.038, 3.10)	0.088
HZ	(5, 86), 15.2	(0.038, 2.38)	0.543

Modeling cases

The comparison presented herein comprises three cases, as described in Table 4. In these cases, the numbers of fractures are set to 10, 100, and 1 000, respectively. The DFN is bounded by a $200 \times 200 \times 200$ m region, with a constant-head boundary of 20 m on the panel at $x = -100$ m and a constant-head boundary of 10 m on the panel at $x = 100$ m. On the other side, the edges of the fractures act as no-flow boundaries; note that some fractures extend beyond the region boundary. The flow is simulated under steady-state conditions.

Table 4. Properties of comparison cases.

No. of fractures	No. of intersections	No. of vertices FracMan®	No. of elements AnDFN
10	19	15 662	33
100	314	61 813	449
1 000	2 063	205 453	3 173

Assessment of AnDFN model accuracy

The boundary-condition residuals of the AnDFN model for the three cases are presented in Table 5. The model was run with 200 coefficients for each element. The small residuals indicate that the governing boundary conditions are satisfied to a high degree of precision and therefore provide a measure of the numerical accuracy of the AnDFN solution. Comparisons with other models can then be used to assess the consistency of the respective solutions for the considered test cases.

Table 5. AnDFN boundary-condition residuals for the comparison cases.

No. of fractures	max	99th percentile	95th percentile
10	0.014%	0.014%	0.011%
100	0.007%	0.004%	0.002%
1 000	0.964%	0.007%	0.003%

Hydraulic head

Figure 8 presents the hydraulic head contour for the 1 000-fracture case, simulated using AnDFN. Overall, the hydraulic head values gradually vary between the inlet boundary (where the hydraulic head is specified as 20 m) and the outlet boundary (where the hydraulic head is 10 m).

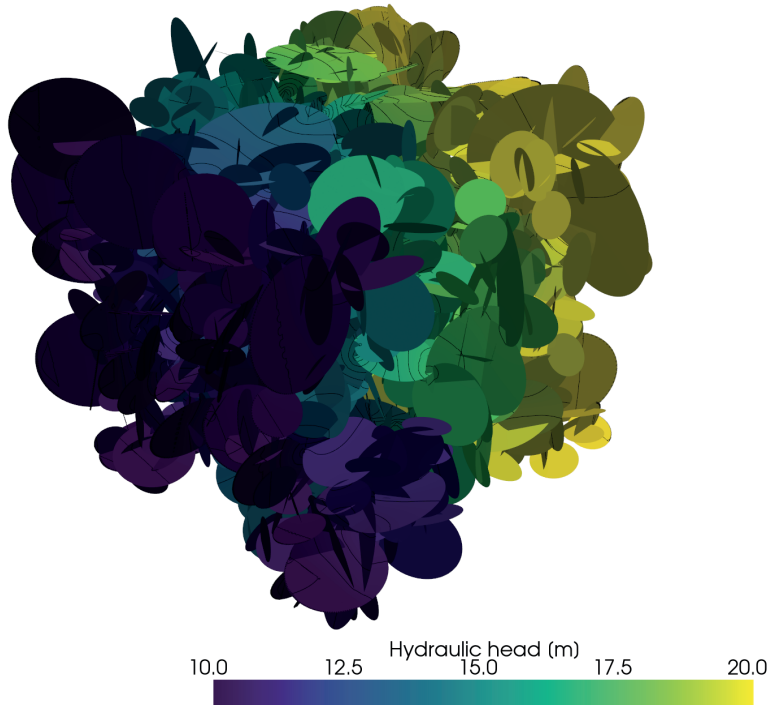


Figure 8. Hydraulic head contours for the 1 000-fracture case; the black lines represent equipotential lines.

The comparison of hydraulic head values obtained from AnDFN and FracMan® is presented in Figure 9 for all three cases. Specifically, the comparison is performed by evalu-

ating the hydraulic head at the node locations of the FracMan[®] mesh, which has a resolution of 1 m. Since AnDFN does not have nodes, the corresponding values are calculated at the coordinates of these nodes. For all cases, all nodes within the models are included in the comparison. The results show that, for these three cases, the hydraulic heads at the nodes are generally in agreement between AnDFN and FracMan[®].

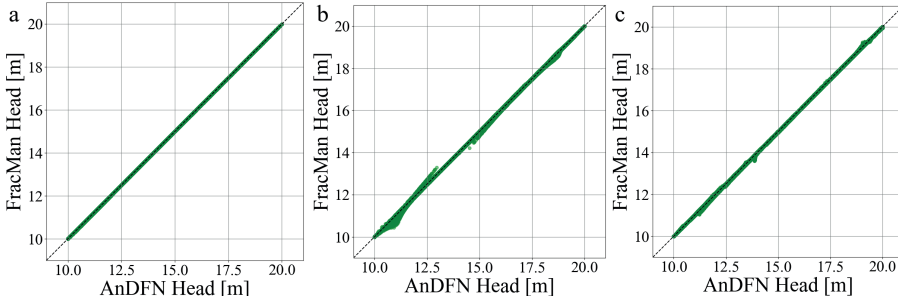


Figure 9. Comparison of the hydraulic head for DFNs with 10 (a), 100 (b), and 1 000 (c) fractures.

Figure 10 further presents the histogram of the relative difference between the simulated hydraulic head from AnDFN and FracMan[®]. The results show that the relative differences for the vast majority of points are consistently small and close to zero for all three cases, indicating strong agreement between the two models.

The largest differences occur at small intersections (maximum values of 0.42% for the 10-fracture case, 4.46% for the 100-fracture case, and 2.02% for the 1 000-fracture case). The larger difference in the 100-fracture case arises because there are two instances where the FracMan[®] model identifies intersections differently from AnDFN. This discrepancy is due to FracMan[®] representing the circles as 64-sided polygons. First, there is a location where the FracMan[®] model identifies an intersection that is not identified by AnDFN. Additionally, there is another location in the 100-fracture case where the length of the intersection is 0.125 m, while the radii of the corresponding fractures are 47.0 m and 45.7 m. Given that the mesh size is 1 m, the mesh is likely unable to resolve such small intersections accurately. Nonetheless, Figure 10 shows that this effect is very local, has only a minor impact on the overall head distribution, and that the difference is due to a geometrical representation difference rather than a flow solution difference.

Particle tracking

In FracMan[®], particle tracking is conducted based on the underlying connections between control volumes (cells), where particles are tracked through each volume according to the edge flux rather than a fully resolved two-dimensional velocity field within each fracture, as can be calculated with AnDFN. AnDFN provides continuous velocity fields for particle tracking without the need for interpolation from a mesh. For comparison, particle-tracking

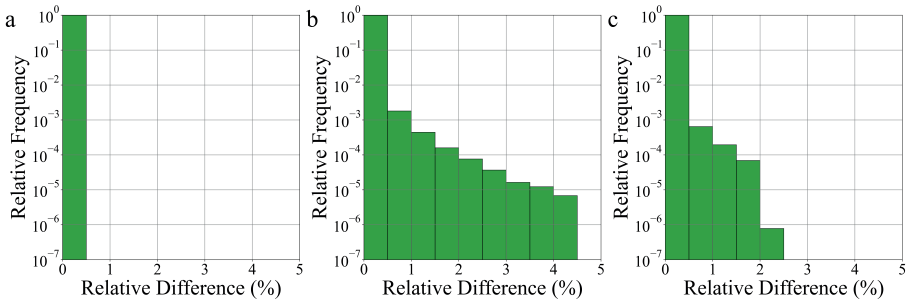


Figure 10. Histogram of the relative hydraulic-head differences for DFNs with 10 (a), 100 (b), and 1 000 (c) fractures.

results for the 1 000-fracture case from FracMan® and AnDFN are presented. Note that the results are not expected to match exactly due to the difference in approach.

Figure 11 presents a comparison of the trace length distribution between AnDFN and FracMan®. In FracMan®, 500 particles were released based on flow weighting at the fracture with the highest inflow. These starting positions were extracted and used in AnDFN. The trace length simulated by AnDFN is generally shorter than that simulated by FracMan®. Such differences are expected when comparing continuous trajectories with trajectories computed on a discretized mesh.

Figure 12 shows a comparison of the arrival time distributions simulated by both FracMan® and AnDFN. The travel times range from 0.1 to 6 years. For a given arrival time, the cumulative arrival fractions predicted by AnDFN are approximately two to three times larger than those predicted by FracMan®. Although trace lengths in FracMan® are longer, as particles can only travel between cell centers, FracMan® adjusts particle travel times to account for the fact that particle paths are not locally aligned with the flow direction. This adjustment reduces the predicted arrival times.

It is important to note that AnDFN traces streamlines, while FracMan® assesses cell jumping (Bym, 2023), which leads to particle tracks that cross equipotential lines at non-perpendicular angles. As a result, AnDFN and FracMan® do not track identical particle paths.

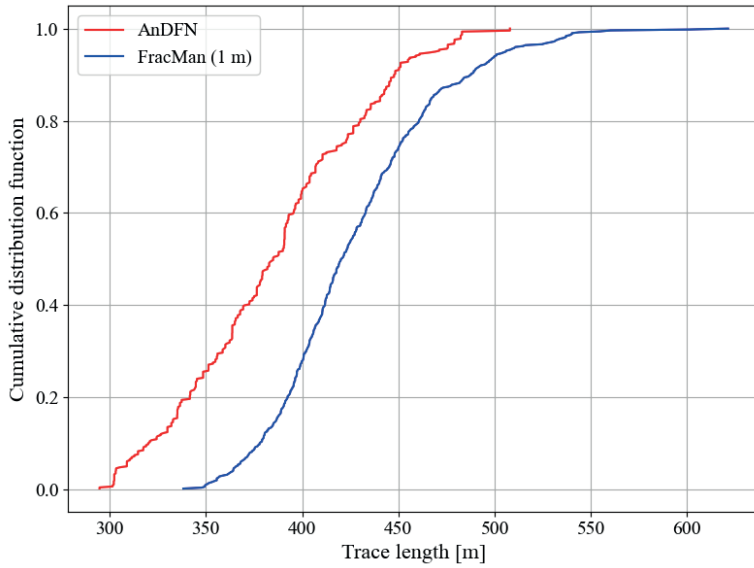


Figure 11. CDF of trace length for particles released at the inflow boundary and traced to the outlet boundary for AnDFN and FracMan®.

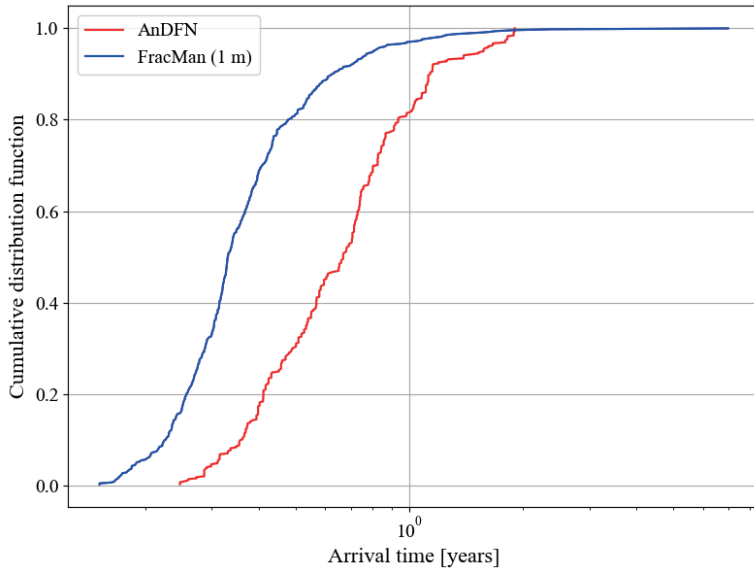


Figure 12. CDF of arrival time for particles released at the inflow boundary and traced to the outlet boundary for AnDFN and FracMan®.

4 APPLICATIONS

This chapter demonstrates the applicability of AnDFN to a range of practical groundwater modeling problems in fractured rock. The model is used to investigate tunnel inflow, derive equivalent continuum properties through hydraulic upscaling, and simulate coupled DFN–continuum flow systems.

4.1 Integration between AnDFN and DarcyTools

The integration of AnDFN with DarcyTools makes it possible to replace part of the DarcyTools model with a DFN. A simple model, in which the central region is replaced by a DFN, is studied here. This model has dimensions of $600 \times 600 \times 200$ m, while the DFN is $200 \times 200 \times 200$ m. The DFN consists of 1 000 fractures and is identical to the network used in the verification study. The dimensions and layout of the modeling cases are presented in Figure 13. All fractures that intersect the DarcyTools domain are assigned a constant-head element, i.e., the two models are linked on all four sides of the DFN. The DarcyTools model has a homogeneous permeability of $8.15 \cdot 10^{-14} \text{ m}^2$ and a porosity of 0.01. The density of water is set to $1\,000 \text{ kg/m}^3$, the standard acceleration due to gravity to 9.81 N/kg , and the viscosity to $0.001 \text{ Pa}\cdot\text{s}$.

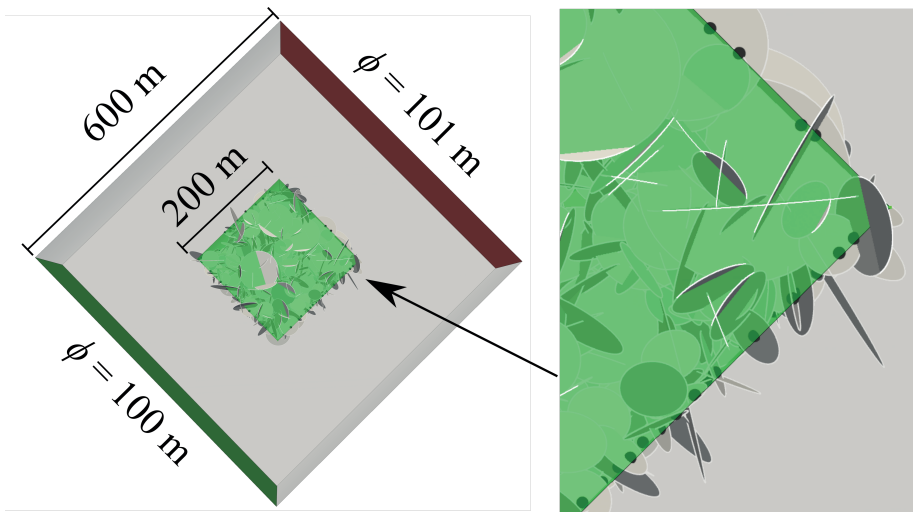


Figure 13. Top view of the geometry and the interface of the AnDFN and DarcyTools model. The red and dark green panels correspond to constant-head boundaries, the light green box is the region that is clipped in DarcyTools and where the DFN is inserted, the light gray disks are the fractures, and the small black dots (on the boundary of the DFN region) are the spot boundary condition locations in DarcyTools. The insert on the right is a zoomed version of the right corner of the DFN region.

From Figure 13, it can be seen that the fractures extend into the DarcyTools model, as expected. However, the constant-head line in the fracture that forms the interface connection lies along the interface between the two models. In this model, fractures are generated from their center points, meaning that at most half of each fracture extends into the ECM. This leads to an underestimation of connectivity between the ECM and the DFN. On the other hand, it avoids introducing artificially large portions of fractures within the ECM.

Since each model is solved independently, an iterative scheme is used (see Figure 4) to couple them. The simulations are repeated until convergence is reached, such that continuity of hydraulic head and discharge is satisfied to a given tolerance of 2% in this example. The difference in hydraulic head and flux at the interface boundary, i.e., the error, is initially large but gradually decreases, as illustrated in Figure 14. For this example (see Figure 13), the tolerance was met after 12 iterations. It is important to note that this is a proof-of-concept, and more rigorous testing is required for the method to be applicable in a realistic scenario.

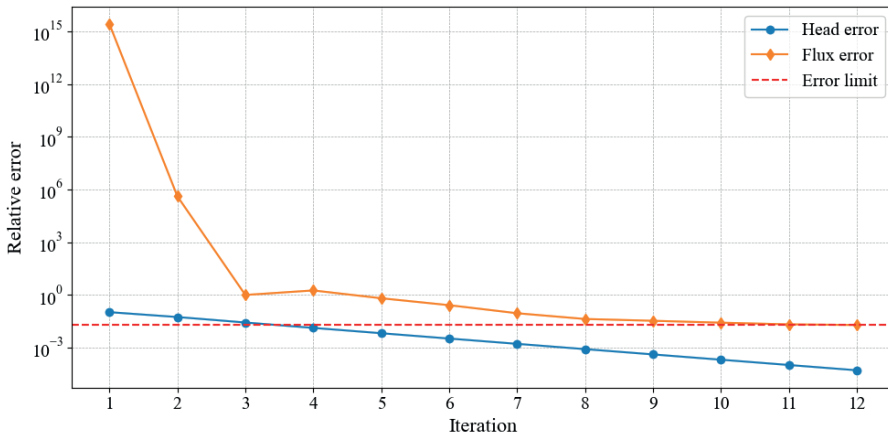


Figure 14. Reduction in hydraulic-head and flux errors over successive iterations for the model presented in Figure 13.

The steady-state solution is presented in Figure 15. It can be seen that the hydraulic head distribution in the DFN closely resembles that in the ECM, which is to be expected since the DFN is coupled on all sides and thus inherits its hydraulic gradient directly from the surrounding ECM. Consequently, the overall flow pattern within the DFN follows the same large-scale trend as in the ECM, with only local deviations caused by the discrete fracture geometry and their connectivity.

Because the particle-tracking routine is part of the post-processing of both models, it is possible to perform particle tracking continuously across both models. The particle tracking is implemented as a separate script developed in this study, which reads the final flow

solutions from both DarcyTools and AnDFN. At this stage, the flow field is assumed to have reached steady-state. Once a particle exits at the interface boundary in one of the models, its location is transferred to the other model, and the particle tracking continues. Note that, even though the fractures extend into the DarcyTools model, the interface line is located along the interface between the two models, and this is where particles transition from one model to the other.

Figure 15 also shows particle tracks extending both backward and forward from the center point of all fractures. The results show that the capture zone is limited to the central portion of the ECM constant-head boundary. This indicates that the effective permeability of the DFN is lower than that of the surrounding ECM, as the majority of the flow is directed around the DFN through the ECM portion of the model. It is important to note that flow within the DFN is driven entirely by the hydraulic-head distribution in the surrounding ECM.

The idea of replacing only a part of the model with a DFN is that high resolution can be achieved around a specific object of interest without increasing the mesh resolution of the entire model, thereby reducing the computational cost. This principle holds for any combined DFN-ECM model and is not limited to the model used in this study. To demonstrate this, a DarcyTools model using the same set-up as previously is extended by introducing a fully penetrating six-sided well with atmospheric pressure located at the center of the DFN. Figure 16 shows the resulting steady-state head distribution, with a zoomed-in view presented in the inserted panel. From the well, 366 particles were released and backward-tracked through the DFN and ECM; these tracks are also displayed in Figure 16. The particles were released at every constant-head boundary enforced by the well, i.e., they are evenly distributed along the well. The particle tracks shown in Figure 16 suggest that most of the capture zone extends toward the inflow boundary, while only a limited portion originates from the outflow boundary. Due to the heterogeneity of the DFN, the elevation of the sources is distributed throughout the ECM.

The examples presented here assume an ECM with isotropic permeability, resulting in a high contrast between the homogeneous permeability of the ECM and the fracture transmissivity of the DFN. This high contrast can give rise to numerical noise due to floating-point precision and makes the model sensitive to small changes in flux. The large contrast in effective permeability between the DFN and ECM also causes some of the flow to bypass the DFN. Integrating the two solvers directly would likely reduce this effect.

Using an upscaled anisotropic permeability field would further improve the model. In particular, such an approach would allow the ECM to capture the preferential flow behavior imposed by the DFN, leading to a more consistent hydraulic representation. However, this does not eliminate the fundamental differences in representation of the two models: explicit representation in the DFN and averaged representation in the ECM. Consequently, some degree of mismatch between the DFN and ECM is unavoidable. If the DFN and ECM are constructed using different data sources, for example, pump tests for the ECM and conditioned fractures for the DFN, additional inconsistencies are introduced.

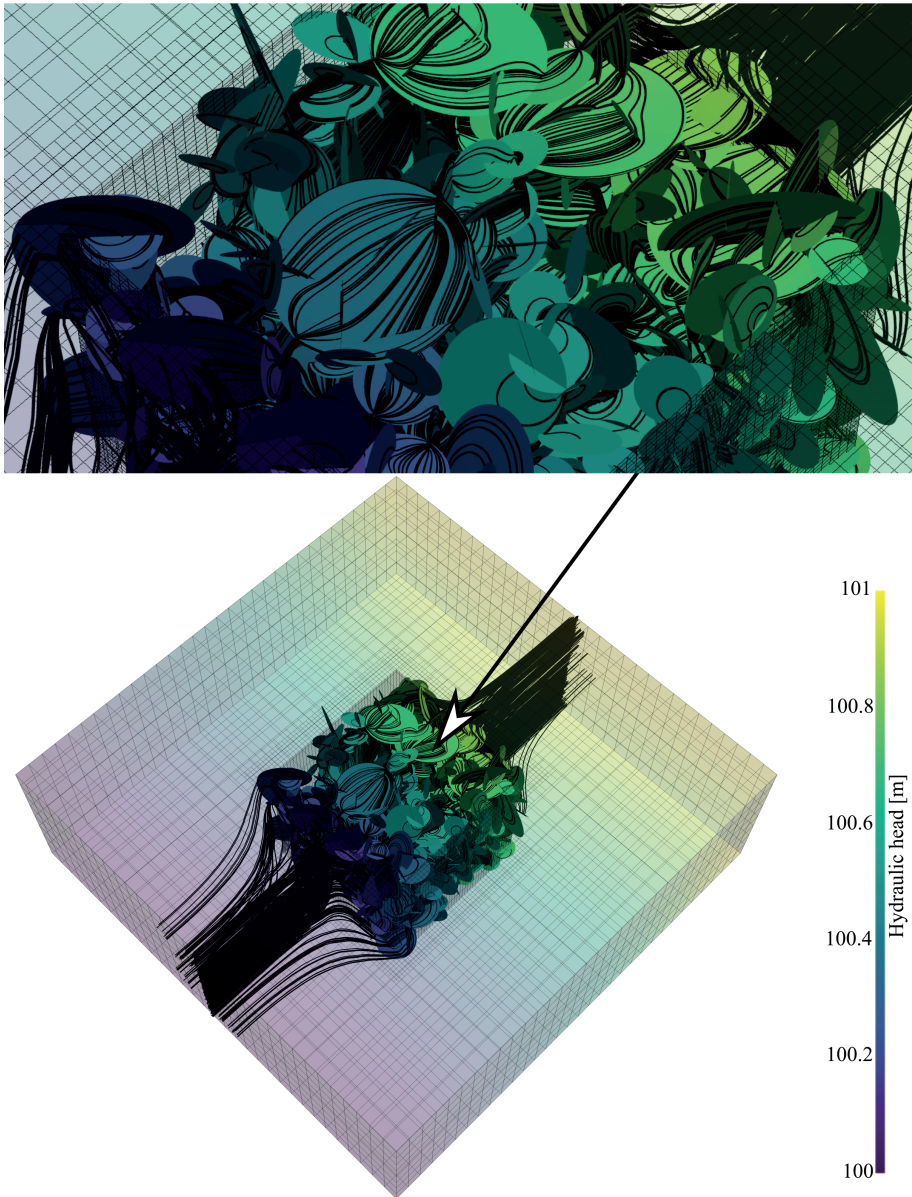


Figure 15. Hydraulic head contours for the model shown in Figure 13, with particles tracked both forward and backward from the centers of all fractures; a zoomed-in view of the DFN is shown in the inset panel.

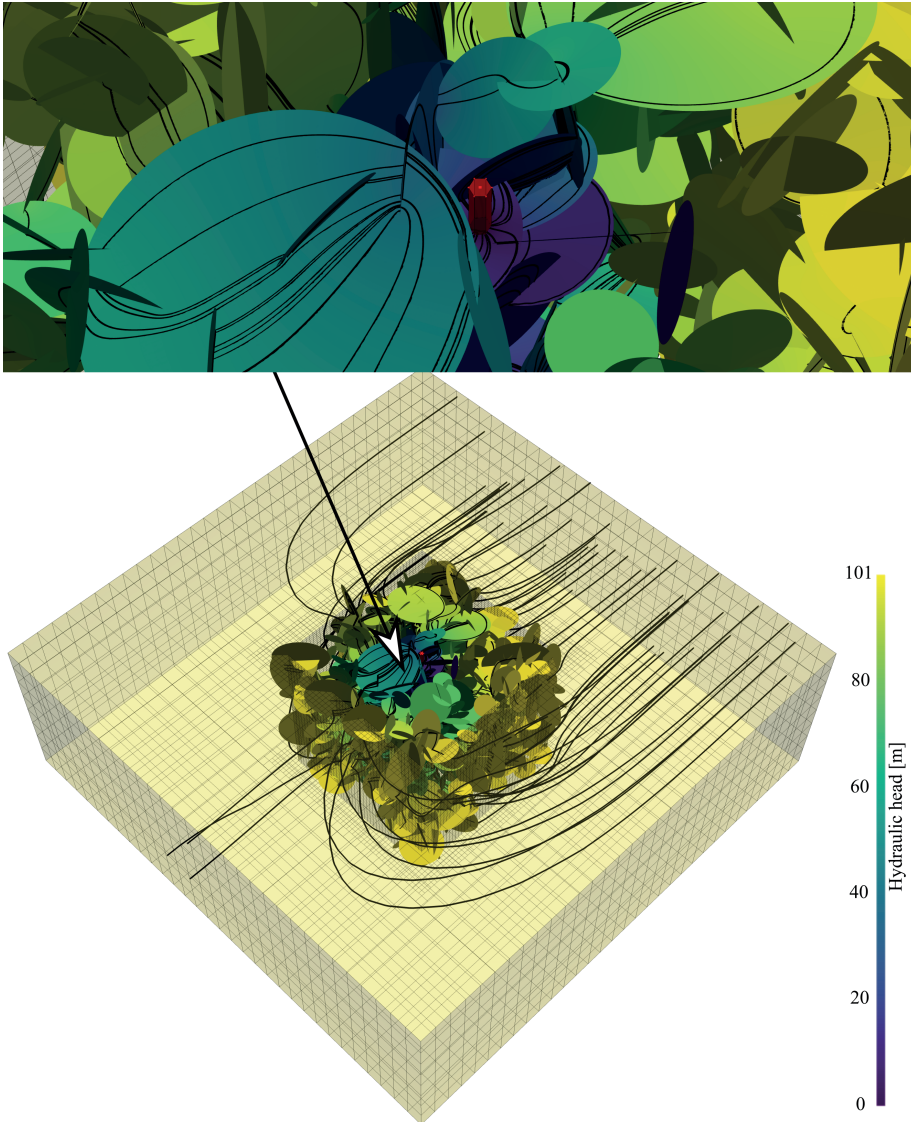


Figure 16. Hydraulic head contours for a model with a pumping well located at the center of the DFN, with 366 particles tracked backward; a zoomed-in view of the DFN is shown in the inset panel.

4.2 Tunnel inflow

The next application concerns the simulation of tunnel inflow using the developed DFN model. In this example, the model domain is a $1\,000 \times 500 \times 250$ m box with its center at the origin and an 8-sided tunnel with a 5 m radius extending from (0, -250, -20) to (0, 250, -20). Within the model domain, fractures are generated using the parameters given in Table 3, with a minimum radius of 8 m and a maximum of 150 m. Around the tunnel, additional smaller fractures are then generated using the same fracture sets but with a minimum radius of 1 m and a maximum of 25 m. The tunnel has an atmospheric boundary condition, and the top surface has a fixed constant head. The first realization and its boundary conditions are presented in Figure 17. To investigate the natural variability introduced by the DFN on the tunnel inflow, 50 realizations were run.

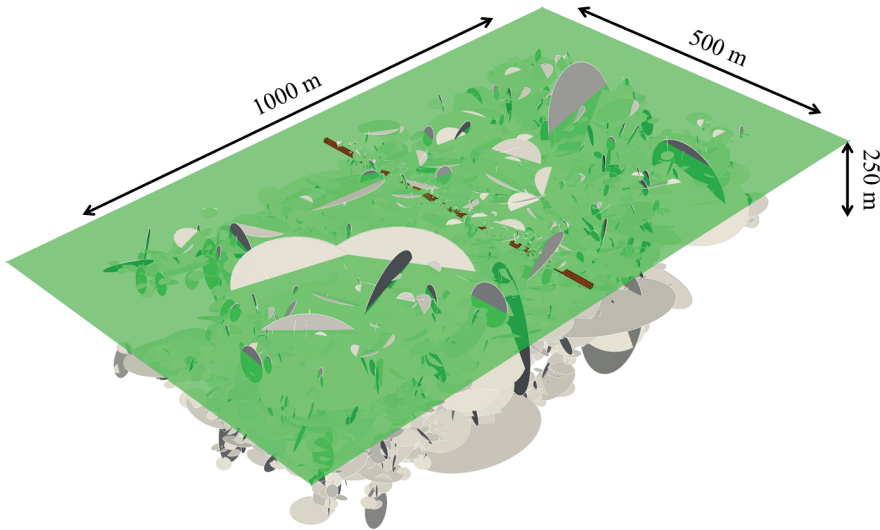


Figure 17. The first realization of fractures with the tunnel in red and the top surface boundary in green. Note that the surface boundary extends beyond the DFN to avoid the terminations of the constant-head boundary condition within a fracture.

Baseline tunnel inflow

Figure 18 presents the hydraulic head distribution for the first realization. It can be observed that some fractures intersect both the tunnel and the surface, resulting in steep hydraulic head gradients along these fractures. Nevertheless, the influence of the tunnel extends through the DFN and remains visible at distances of up to 400 m. At depth, the impact of the tunnel extends throughout nearly the entire model domain.

One advantage of using a DFN for tunnel inflow estimation is that it allows the inflow

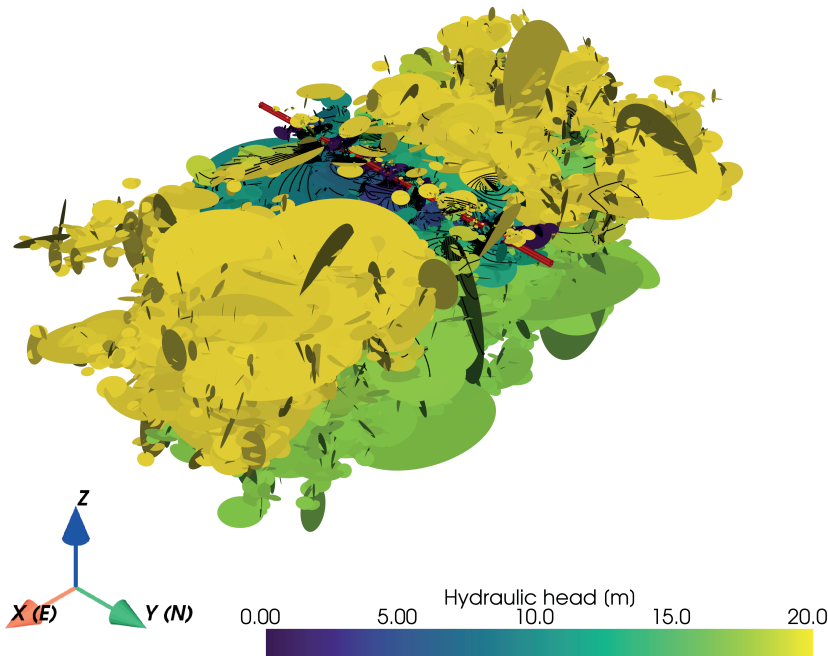


Figure 18. The hydraulic head distribution with equipotential lines (black lines) for the first realization.

distribution along the tunnel to be computed, thereby providing insight into its natural variability. Figure 19 shows only the hydraulic head for the fractures intersecting the tunnel in the first realization. The hydraulic gradient in these fractures varies significantly depending on their connectivity to the surface boundary.

Figure 20 presents the inflow along the tunnel together with the mean and maximum transmissivity of the intersecting fractures for the first realization. In this plot, the tunnel is divided into 1 m segments, for which the mean inflow is calculated for each segment, along with the mean and maximum transmissivity of all fractures intersecting that segment. From Figure 20, it is straightforward to identify where the highest inflows occur and to observe that these are not necessarily directly correlated with transmissivity. Figure 21 presents the mean inflow for all 50 realizations, aggregated into 100 m segments. With the exception of the first and last segments, where edge effects are present, the mean inflow varies between 3 and 5 l/min per 100 m, indicating that, on average, the inflow along the tunnel is relatively stable. Within each 100 m segment, however, the inflow is likely to be highly heterogeneous.

The total tunnel inflow for each realization is presented in Figure 22. The results show a considerable variability in tunnel inflow between realizations, ranging from approximately 2 to 65 l/min. The mean total inflow is about 18 l/min, while the 5th and 95th percentiles

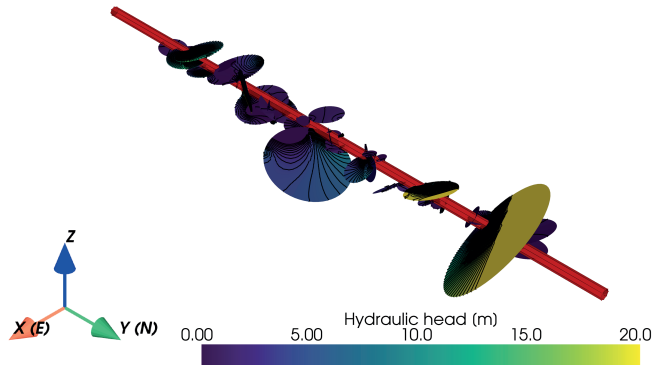


Figure 19. The hydraulic head distribution for the fractures intersecting the tunnel, with equipotential lines (black lines), for the first realization.

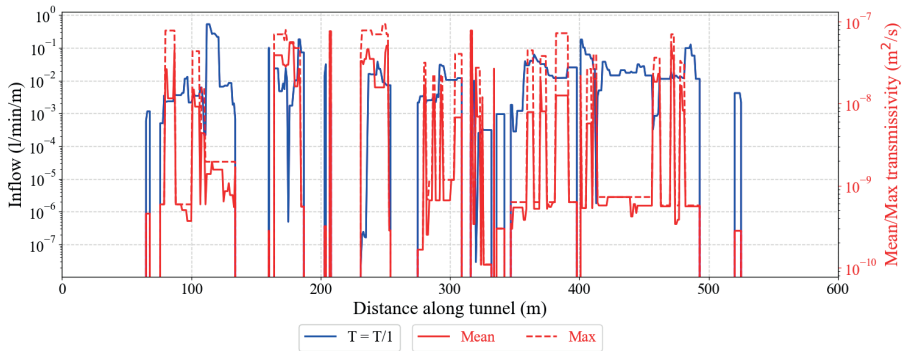


Figure 20. The tunnel inflow and the mean and maximum transmissivity per meter of tunnel length for the first realization.

are approximately 5 and 37 l/min, respectively. Most realizations yield total inflows below 20 l/min, although several higher inflow realizations are observed. One realization yields a total inflow of around 65 l/min, indicating the presence of a particularly conductive fracture network configuration. The spread of results highlights the sensitivity of tunnel inflow to the stochastic representation of the hydraulic properties and fracture distribution.

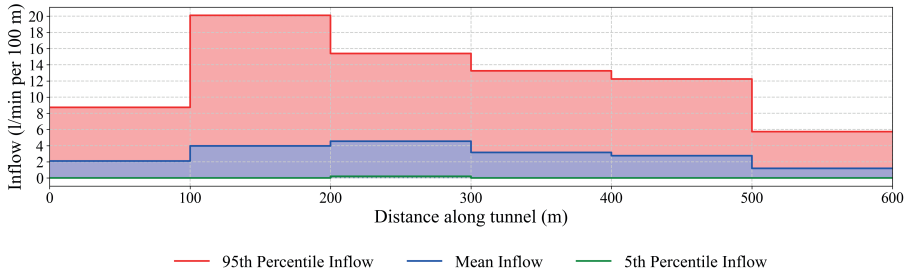


Figure 21. The mean, 95th percentile, and 5th percentile of tunnel inflow per 100 m segments for all 50 realizations.

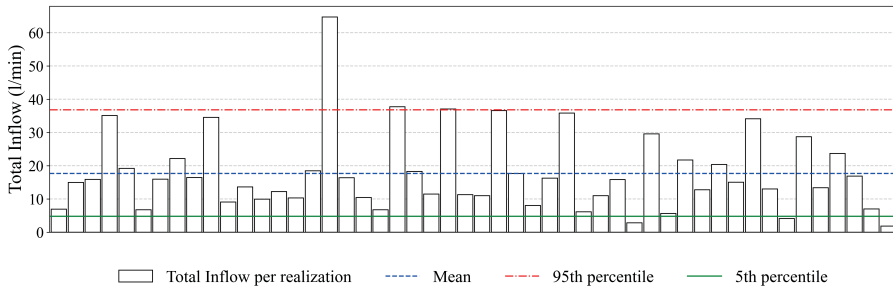


Figure 22. The total inflow for the tunnel, with lines for mean, 95th percentile, and 5th percentile.

Transmissivity reduction

The next case studies concern a reduction in transmissivity and its influence on the tunnel inflow. The transmissivity was reduced for two different cases:

1. the transmissivity is gradually decreased for all fractures that intersect the tunnel, hereafter denoted as “all”,
2. the transmissivity is gradually decreased for all fractures that intersect the tunnel and have a transmissivity greater than 10^{-7} m/s, hereafter denoted as “ $T > 10^{-7}$ ”.

The reductions were applied using the following factors: $T/2.5$, $T/5$, $T/7.5$, $T/10$, $T/25$, $T/50$, $T/75$, $T/100$, $T/250$, $T/500$, $T/750$, and $T/1000$, where T is the initial transmissivity.

Figure 23 presents the inflow for all transmissivity reductions for the first realization in the “all” case; the inflow is calculated in the same manner as in Figure 20. Overall, the inflow decreases with reduced transmissivity. However, at some locations, the inflow initially increases as transmissivity is reduced. This occurs because the hydraulic gradient increases

in certain fractures when inflow in others is reduced, leading to locally higher inflow. As the transmissivity is reduced further, the inflow in these fractures also decreases.

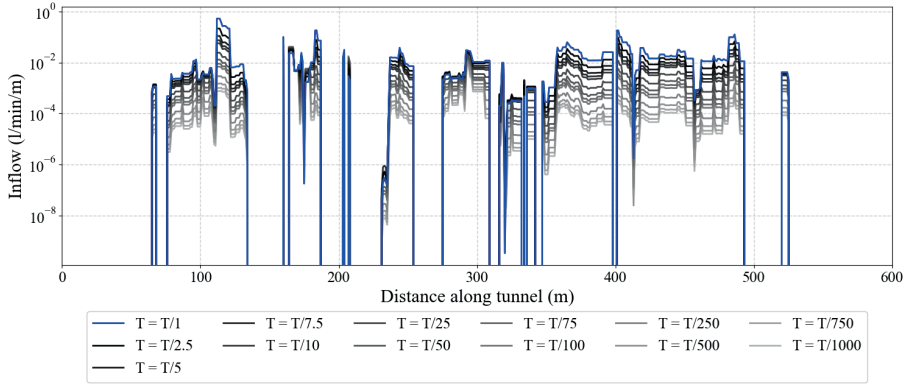


Figure 23. The tunnel inflow per meter of tunnel length for different transmissivity reductions in the first realization for the "all" case.

Comparing the results for all 50 realizations, the reduction in inflow can vary significantly. Figure 24 presents the distribution of the remaining inflow for a selection of transmissivity reductions for the "all" case and the " $T > 10^{-7}$ " case. The remaining inflow is defined as the computed total inflow divided by the initial total inflow. For small reductions in transmissivity, both cases show a similar variation in remaining inflow. However, as the transmissivity is reduced further, the " $T > 10^{-7}$ " case shows a much wider spread in the remaining inflow.

Although the DFN is relatively small and the influence of the low-transmissivity fractures is limited, it still provides valuable insight. The results demonstrate that fractures can have a significant impact on tunnel inflow and that predicting inflow is not as straightforward as assuming that the largest fractures exert the greatest influence. Furthermore, if grouting fails to reduce the transmissivity of the smaller fracture, a portion of the total inflow will remain unavoidable, as groundwater will continue to enter the tunnel through this hydraulically conductive pathway regardless of the effectiveness of grouting in the surrounding fractures. The study could also be extended to identify, for example, problematic fracture orientations or tunnel alignments that are particularly susceptible to high inflow rates.

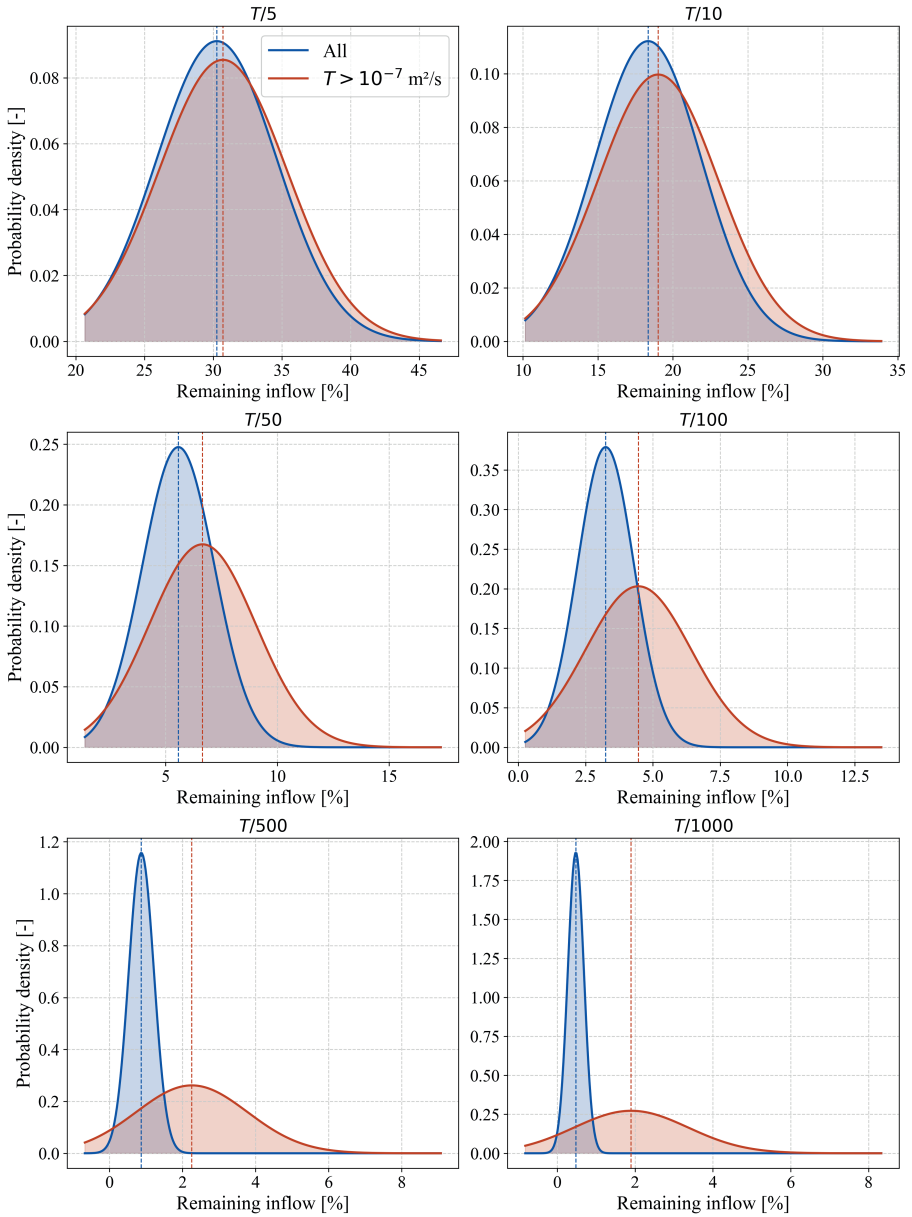


Figure 24. Normal distributions of remaining inflow at selected transmissivity factors ($T/5$, $T/10$, $T/50$, $T/100$, $T/500$, $T/1000$); blue lines represent the "all" case and red lines represent the " $T > 10^{-7}$ " case.

4.3 Hydraulic upscaling using discrete fracture networks

In many cases, it is infeasible to use a DFN, and a continuum model is preferable. One method to incorporate DFN properties into a continuum model is to upscale the DFN. As a complementary study to this research, two master's thesis projects were conducted, focusing on hydraulic upscaling of a DFN. These projects are published in Marinelli (2026) and Berg and Blomqvist (2026). Here, only a brief summary of these studies is included.

Both Marinelli (2026) and Berg and Blomqvist (2026) used the same upscaling approach and the same set of 12 DFN realizations. The DFNs were generated in a $1\,000 \times 1\,000 \times 1\,000$ m box with the fracture sets presented in Table 3. For a DFN to be appropriate as an ECM, the computed hydraulic conductivity tensor should exhibit an approximately ellipsoidal, or ideally spherical, shape. In Berg and Blomqvist (2026), the upscaled hydraulic conductivity tensor was used to evaluate the applicability of ECMs and a simple analytical function for tunnel inflow analysis, while Marinelli (2026) combined the hydraulic conductivity tensor with an upscaled porosity tensor to evaluate the performance of an ECM for particle tracking.

To upscale the DFN, a control volume was defined at the center of the DFN, measuring $500 \times 500 \times 500$ m, with its own local coordinate system $\tilde{x}, \tilde{y}, \tilde{z}$. A hydraulic difference was applied in one principal direction at a time, and the total flow was recorded. The control volume was then rotated 15° about the $\tilde{z}$ axis, and the flow over $\tilde{x}$ and $\tilde{y}$ was computed again. After five 15° rotations about the $\tilde{z}$ axis, the control volume was rotated 15° about the $\tilde{x}$ axis, and the flow was recomputed. The rotation continued until the flow in all directions at 15° increments had been computed. This resulted in 266 simulations for each realization; for 12 realizations, this amounts to 3 192 simulations in total. Each DFN realization initially contained approximately 150 000 fractures, of which approximately 80 000 were connected to a boundary and approximately 9 500 were located within the control volume, depending on orientation.

Based on the flow for each direction, the resulting equivalent hydraulic conductivity was computed using

$$k_i = \frac{Q_i L}{\Delta \phi A} \quad (28)$$

where Q_i is the flow through the DFN, L is the length (500 m), $\Delta \phi$ is the difference in hydraulic head, and A is the cross-section area ($250\,000 \text{ m}^2$). The 266 resulting points were plotted in three-dimensional space, and an ellipsoid was fitted using a least-squares fitting approach. A shell of the resulting conductivity values for all realizations is presented in Figure 25. In addition to visual inspection of how well the plotted tensor conforms to an ellipsoid, Wang et al. (2023) proposed using the normalized relative error

$$E_r = \frac{1}{N} \sum_{i=1}^N \left| 1 - \frac{k_{\text{measured},i}}{k_{\text{predicted},i}} \right| \quad (29)$$

where $k_{\text{measured},i}$ are the measured directional conductivity values in direction i and $k_{\text{predicted},i}$ are the values predicted by the fitted tensor in the same direction. Wang et al.

(2023) suggested an acceptance threshold of $E_r < 0.3$. Out of the 12 DFN realizations, 7 were below this threshold.

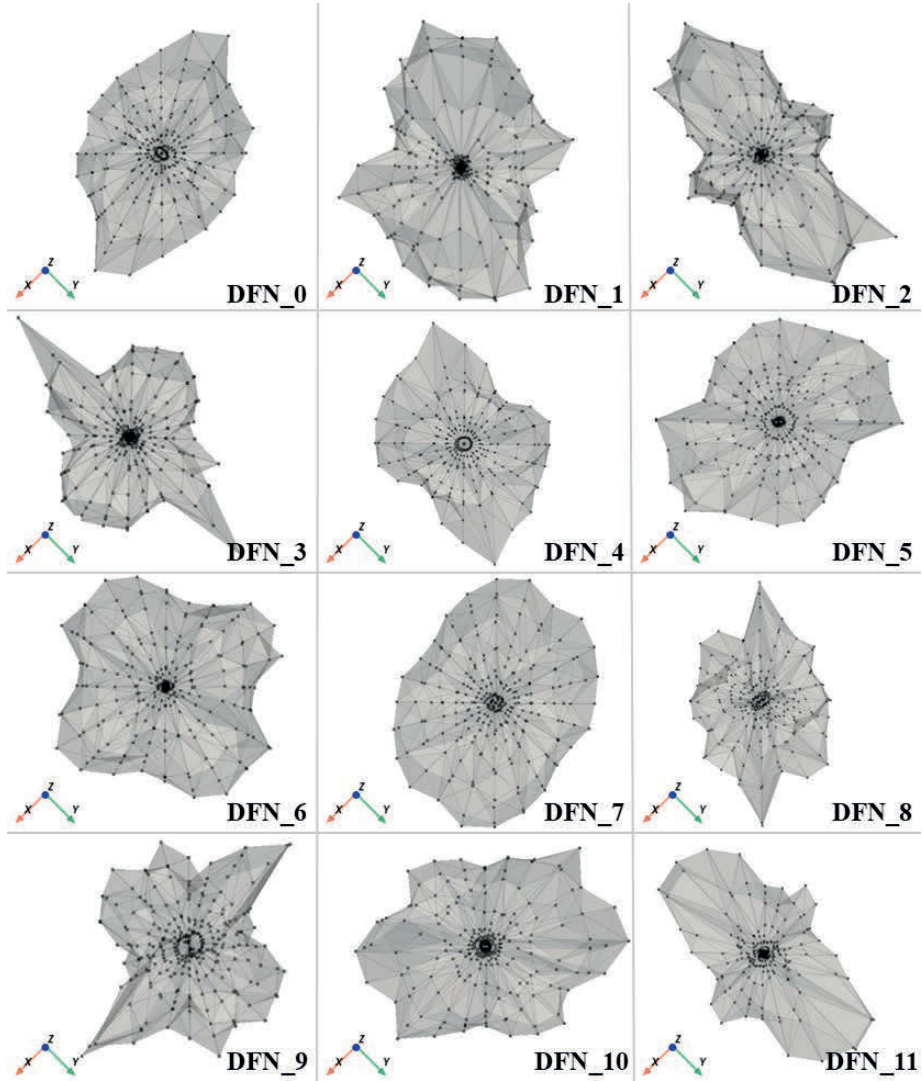


Figure 25. Top view of the hydraulic conductivity values for all twelve realizations (from Berg and Blomqvist (2026) with permission).

As part of a tunnel inflow scenario, the resulting effective hydraulic conductivity tensor was later implemented in a FloPy/MODFLOW model; for more information on FloPy, see Bakker et al. (2016) and Hughes et al. (2024). The two models are displayed in Figure 26.

The tunnel is located at a depth of 100 m and has an atmospheric boundary condition, while the top surface has a fixed constant head.

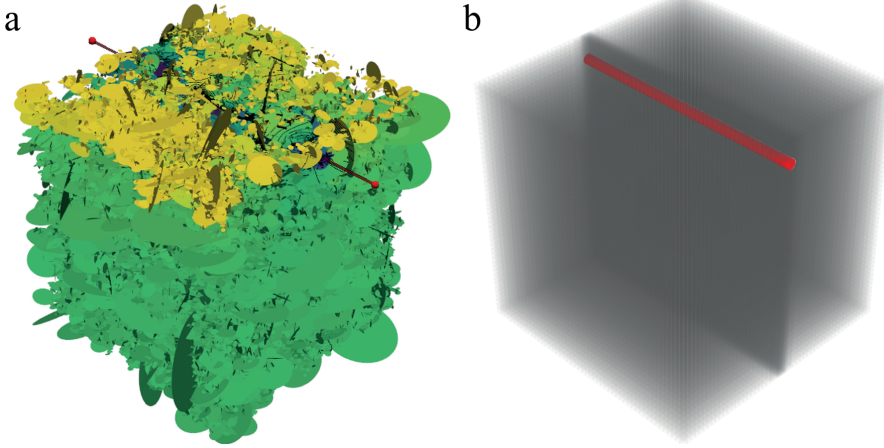


Figure 26. (a) One of the DFN realizations with the tunnel and (b) the corresponding FloPy mesh with the tunnel (both figures from Berg and Blomqvist (2026) with permission).

The results of the two models were compared to the Goodman equation for tunnel inflow:

$$Q_t = 2\pi k_{\text{avg},1,3} \frac{\phi}{\ln\left(\frac{2\zeta}{\tilde{r}}\right)} \quad (30)$$

where Q_t is the tunnel inflow, $k_{\text{avg},1,3}$ is either the average hydraulic conductivity from the upscaled tensor (k_{avg}), the major principal value of the upscaled tensor (k_1), or the minor principal value of the upscaled tensor (k_3), ζ is the distance between the tunnel and the constant groundwater surface, and $\tilde{r}$ is the radius of the circular tunnel (Goodman et al., 1964). The results are presented in Figure 27. Note that the major and minor principal directions do not necessarily coincide with the Cartesian axes; however, in all but two realizations, k_3 is, in a practical sense, parallel to the z -axis, with the exceptions being DFN_8 and DFN_9.

The results presented in Figure 27 indicate that using the Goodman equation with the average hydraulic conductivity from the tensor yields results similar to those of the DFN, while the other approaches, including FloPy, differ significantly. For the Goodman equation, using k_1 or k_3 results in either overestimation or underestimation, which is expected since neither the highest nor the lowest tensor component is representative of the average hydraulic properties of the bedrock. For FloPy, the difference relative to the DFN is on average 68%. This large discrepancy is due to high anisotropy, with a very low vertical hydraulic conductivity, which is overestimated in the continuum model. Although

the effective hydraulic conductivity is correct on average, the DFN contains fractures that intersect both the surface and the tunnel, thereby bypassing the vertical resistance.

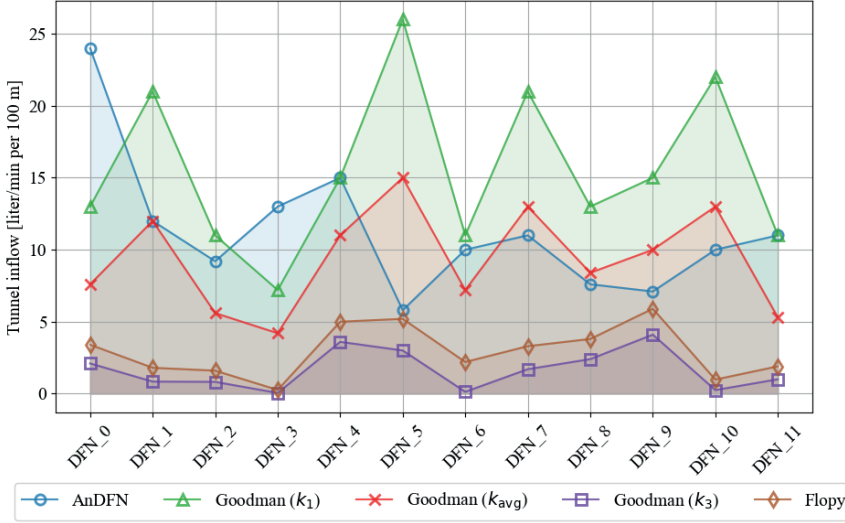


Figure 27. Tunnel inflow using the three modeling methods for the different realizations.

In addition to the tunnel inflow scenario by Berg and Blomqvist (2026), Marinelli (2026) conducted porosity upscaling for the DFN realization with the lowest E_r value. The porosity was upscaled in the direction of the three principal components of the upscaled hydraulic conductivity tensor. The upscaled porosity was calculated kinematically by measuring the travel time for particles to move through the DFN from the inflow panel to the outflow panel. The porosity was then calculated using

$$n_{\text{fast}} = \frac{Q_p t_{\text{fast}}}{V} \quad \text{and} \quad n_{\text{median}} = \frac{Q_p t_{\text{median}}}{V} \quad (31)$$

where n is the porosity, Q_p is the total panel inflow, t is time, and V is the volume of the control volume. The subscripts “fast” and “median” denote whether the first arrival time or the median travel time is used. In this study, only advective transport is considered.

The results were first verified in a FloPy model, where the same travel times were reproduced using the upscaled hydraulic conductivity tensor and the upscaled porosity. The two models also showed good agreement in total flow through the volume. Subsequently, a test scenario was simulated for a pumping well. The 100 m long pumping well, with a radius of 0.3 m, was placed at the center of the control volume, 20 m below the surface. On one panel of the region box, a constant head of 100 m was imposed, while the well has a constant head of 0 m.

Figure 28 presents the particle tracks from the well in both the FloPy model and the AnDFN model. It can clearly be seen that the DFN introduces blind spots and that the spread of the particles is heterogeneous, whereas the ECM shows a more structured and continuous distribution of particles. This difference highlights the effect of explicitly representing discrete fractures, where flow pathways are controlled by the fracture network geometry, leading to preferential flow channels and regions with limited connectivity. In contrast, the ECM representation smooths out these heterogeneities, resulting in a more uniform particle distribution.

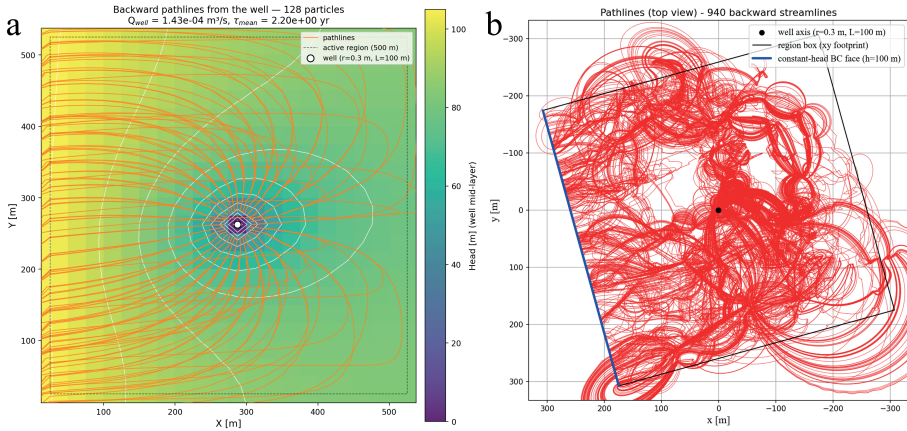


Figure 28. Particle tracks in the ECM (a) and the DFN (b) (from Marinelli (2026) with permission).

In Figure 29, the CDF curves for the travel times are presented. It is evident that the ECM can accurately replicate parts of the distribution; for example, in the median case, the later portion of the arrival times matches well between the ECM and the DFN. For the fast upscaled porosity, the first arrival matches well. This demonstrates that, for this model, an ECM can replicate the arrival-time behavior of a DFN but may, for a single porosity value, capture only a subset of the full distribution curve.

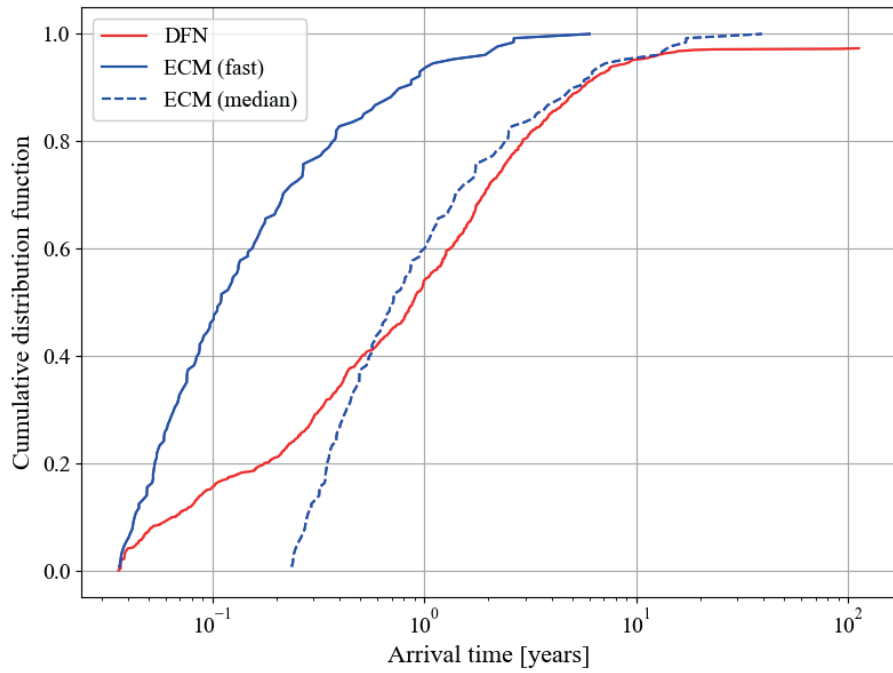


Figure 29. CDF curve of the arrival time of the particles in both the ECM "fast", "median", and the DFN.

5 DISCUSSION AND CONCLUSIONS

This discussion highlights the key findings, advantages, and limitations of the developed analytic DFN model. It is worth noting that a substantial portion of this study focused on the development of the AnDFN model, which is published as an open-source Python package. For in-depth documentation of the code, the reader is referred to the online AnDFN documentation².

The code was developed entirely in Python, which makes it straightforward to combine with other Python packages. This facilitates automatic coupling with other modeling approaches or data analysis tools. It also simplifies the generation of plots, for example, breakthrough curves, as well as the export of data tables and other results. The reliance on Python further supports transparency and reproducibility, as the full workflow can be scripted, version-controlled, and extended with minimal effort.

5.1 An analytic DFN

The formulation of the model enables an analytic representation of a DFN using different series expansions. As a result, the flow field is defined throughout the domain using analytic functions, with a level of precision that can be specified by the user. In practice, this allows the user to balance computational efficiency against accuracy by selecting an appropriate order for the series expansions.

The analytic formulation also provides a quantitative measure of numerical performance through the evaluation of boundary-condition residuals, which quantify how closely the prescribed boundary conditions are satisfied and can be used to assess convergence as the order of the series expansions is increased. Consequently, the solution can serve as a reference against which other numerical approaches may be compared, while also providing a transparent measure of its internal consistency.

This property is particularly relevant for model verification, as it allows other numerical models to be compared against a solution for which the boundary-condition residuals are explicitly quantified. In contrast to traditional benchmark cases, which are often limited to simplified one-dimensional or highly idealized flow scenarios, the present formulation allows verification against more geometrically and hydraulically complex DFN configurations. As such, the approach provides a consistent method for assessing the agreement and convergence behavior of models intended to simulate flow and transport in fractured rock.

A notable property of the developed model is its mesh-free formulation, resulting in continuous hydraulic head and velocity fields throughout the DFN. This has several practical implications. For example, the resolution used for particle tracking can be adjusted during post-processing without requiring local mesh refinement or remeshing. For particle tracking, this removes the need to precompute the flow solution on a fixed grid to ensure accurate trajectories. Instead, the velocity field can be evaluated directly along the particle

²<https://eriktoller.github.io/andfn/>

paths.

5.2 DarcyTools integration

The integration with DarcyTools demonstrates that it is possible to combine the DFN model with an ECM model. This coupling has been verified through a proof-of-concept, showing that the hydraulic head and discharge are consistent between the two models.

A current limitation of the implementation is that each model must be solved separately. This results in substantial data transfer, and the same model must be initialized multiple times. It should be possible to fully integrate the AnDFN model with DarcyTools by modifying the source code in both models. The discharge distribution in AnDFN is solved simultaneously using a matrix solver, which could be incorporated directly into the DarcyTools matrix. Doing so would make it possible to solve all flows simultaneously and ensure that the flow is correctly transferred between the models.

For smaller models, it may still be feasible to use the current integration. For example, it would be possible to include only a few fractures surrounding a well to obtain velocities at fracture–well intersections, while using DarcyTools to handle the far-field flow. This would reduce the number of interface points and thereby decrease the amount of data that needs to be transferred between the models. If the fractures are upscaled using DarcyTools, the flow should be properly represented near the well, and the transfer to the fractures should be smooth.

5.3 Tunnel inflow and upscaling

It has been shown herein that it is possible to simulate tunnel inflow using the analytic DFN model, which provides an inflow value along every fracture–tunnel intersection. However, it is important to note that, in most cases, the DFN is a stochastically generated fracture network based on site-specific fracture set properties. It will therefore not provide an exact representation of reality, and for this reason, multiple realizations are typically needed to collect statistics on which engineering decisions can be based.

The tunnel inflow model demonstrates that it is possible to simulate tunnel inflow for multiple realizations using the developed approach. By fully automating the process in Python, tunnel inflow can be estimated for a large number of realizations. The 50 realizations used in this study required 15–20 minutes to run the simulations and approximately the same amount of time to generate the DFNs. This corresponds to roughly 90 simulations per hour, indicating that computational time is not a limiting factor. Setting up such tunnel inflow models can provide insight into, for example, inflow distributions, grouting effects, problematic fracture orientations, and fracture intensity, and can support grouting design.

The use of the developed model as an upscaling tool was tested by Berg and Blomqvist (2026) and Marinelli (2026). They showed that it is possible, through scripting, to run 266 individual simulations for a single DFN realization and upscale it into an ECM. Berg

and Blomqvist (2026) showed that using the average hydraulic conductivity from such upscaling yields good agreement between the tunnel inflow calculated using Goodman's equation (Goodman et al., 1964) and the DFN. In this particular study, however, using the upscaled tensor in an ECM led to large underestimations of the tunnel inflow.

Marinelli (2026) complemented the upscaled hydraulic conductivity with upscaled porosity and thereby accurately reproduced particle travel times in an ECM for both a uniform flow scenario and flow towards a well. This shows that porosity upscaling has potential and can match parts of the CDF curve. An ECM does not, however, reproduce the full curve, but only parts of it. Furthermore, the spatial distribution of particle tracks does not, as expected, match between the ECM and the DFN, meaning that even though the travel times are reproduced, backward particle tracking for identifying source locations remains limited.

It is important to note that constructing a DFN requires more data than is typically collected in routine practice today. A sufficient amount of data is needed to reconstruct one or multiple fracture sets, including orientations, a fracture size model (e.g., power-law parameters), a fracture intensity model, and a radius–transmissivity relationship. Among these, the size model and the transmissivity relationship are often uncertain in tunneling projects. There is therefore an additional cost associated with geological surveying when constructing a DFN; however, the DFN can be used for applications beyond groundwater simulations, for example, in mechanical engineering.

5.4 Future development

This work marks the initial release of the AnDFN model, and although it is fully functional and capable of simulating groundwater flow, there is still potential for further development. While there is no theoretical limitation on the number of fractures that can be included in the model, additional work remains to test and further develop the implementation for large DFNs. In its current state, it has been tested for up to 250 000 fractures.

Another potential future development involves further refinement of the model for upscaling a DFN into an ECM. In the current implementation, the sides of the control volume are treated as no-flow boundaries, which introduces boundary effects and does not represent a uniform gradient over the volume. The next step would therefore be to develop an analytic element that applies a linearly varying hydraulic head. By doing so, the sides can be assigned a gradient equal to the overall gradient of the control volume, thereby removing this boundary effect.

In its current state, there remains scope for improvement in the solver. One of the current limiting factors is CPU memory. There are several ways to address this, for example, by using compressed sparse column arrays and dynamic preallocation of arrays. These improvements would also enhance parallelization by reducing memory usage, which would further improve computational efficiency.

Another area for future development is the derivation of rigorous upper error bounds for

the AnDFN solution. In its current form, the model evaluates boundary-condition residuals, which provide a quantitative measure of how closely the prescribed boundary conditions are satisfied and can be used to assess numerical convergence. However, these residuals do not provide a rigorous estimate of the solution error throughout the domain. Given the analytic nature of the formulation, there may be an opportunity to establish a mathematical relationship between the boundary-condition residuals and the error in the complex potential. Such a model could enable the derivation of upper bounds on the solution error for a given DFN realization. In addition to improving confidence in model predictions, the availability of rigorous error bounds would strengthen the use of AnDFN as a benchmark and verification tool for other numerical approaches.

Lastly, the model has so far only been tested in synthetic modeling scenarios, and it would be both interesting and valuable to apply it in real-world settings with more complex and variable boundary conditions. Such applications could provide further insight into the robustness of the model, its sensitivity to heterogeneous field conditions, and its ability to capture site-specific behavior. In addition, comparisons with measured data would allow for a more rigorous evaluation of the model's predictive capability and practical applicability.

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