THE PROBABILITY OF FRACTURE OF ELEMENTS CONNECTED IN PARALLEL

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SUMMARY

There are frequently discussions in the literature about the probability of fracture of elements connected in parallel. I have, however, not seen a correct solution of the problem. Some people say that the calculation is very complicated. They are right in the way that the general solution of a large number of elements may be very time consuming, even with the fastest computer. The number of terms in the expression is found to be 2^{n-1} where n is the number of elements. For a small n, on the other hand, it is easy to calculate the probability.

This report shows the solution for 3 elements. It gives the general solution for n elements, and the numerical values for 4 and 5 parallel elements. A comparison is made between serial and parallel connection with up to 10 elements. An approximate solution where the number of terms is highly decreased is described.

INTRODUCTION

Recently an experimental investigation of uniaxial tensile strength of rock materials has been presented /1, 2/. Rather unexpectedly it was shown that this strength parameter appeared to be independent of the sample volume within very large limits, although there was considerable scatter in the strength. In /1/ the experimental results were accompanied by a theoretical derivation of the tensile strength of multiple samples composed of certain elementary samples in various parallel and series arrangements. This derivation is however somewhat erroneous and below a correct analysis is presented. Actually there appears to be other papers on this topic also containing similar errors. Weibull /3/ in an early paper pointed out that the probability of fracture of two elements was in all probability the same in serial and parallel connection. This may be practically true, especially for large m-values.
In a textbook on rock mechanics /4/ Jaeger and Cook extended this idea to the general case with n parallel elements, which must be a mistake.

I will not discuss the details of all these errors but the interested reader is asked to compare the differences in the results on his own. I feel quite confident that my own analysis is correct since it essentially is an expansion of unity (i.e. the probability that something happens) into the probabilities of all possible outcomes of a tensile strength test on the multiple sample arrangement in question.

THREE ELEMENTS IN PARALLEL

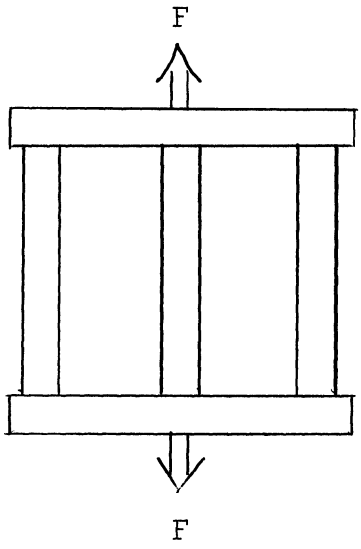

Fig. 1. Three elements connected in parallel.

Consider the case of three elements in parallel shown in figure 1. The total load is always assumed to be equally distributed over the intact elements. The nominal stress σ_o is the stress in each element for the case when all elements are intact. If one element is broken the stress in the remaining two intact elements is $\sigma_1 = 1.5\,\sigma_o$. If two elements are broken the stress in the last intact elements is $3\,\sigma_o$. In the general case of n elements in parallel the stress σ_{n_1} in the remaining $n-n_1$ intact elements is given by

$$\sigma_{n_1} = \frac{n}{n-n_1}\,\sigma_o \qquad (1)$$

We also define the probability function $p(\sigma)$ that one particular element will sustain a stress equal to σ. Accordingly the probability that this particular sample is broken before the stress σ is reached is given by $1 - p(\sigma)$. An example of a suitable analytical expression is given by the Weibull distribtion /3/:

$$p(\sigma) = \exp\left[- (\sigma/\sigma_m)^m \right] \qquad (2)$$

where σ_m and m are parameters.

We now return to the example in figure 1 with three elements in parallel. It is convenient to introduce the probabilities $P_o = p(\sigma_o)$, $P_1 = p(\sigma_1)$ etc, or generally:

$$P_k = p(\sigma_k) \tag{3}$$

Then we can write the probability Q_k that k elements out of the three will be broken by the stress σ_o as:

$$Q_k = \binom{3}{k} (1 - P_o)^k P_o^{3-k} \tag{4}$$

In the general case of n samples in parallel we have:

$$Q_k = \binom{n}{k} (1 - P_o)^k P_o^{n-k} \tag{5}$$

The Q_k in (4) are terms in a binomial expansion as follows:

$$\sum_{k=o}^{3} Q_k = P_o^3 + 3(1-P_o)P_o^2 + 3(1-P_o)^2 P_o + (1-P_o)^3 =$$

$$= [P_o + (1-P_o)]^3 = 1 \tag{6}$$

In (6) the term P_o^3 is the probability that all elements remain intact, the next term that only one element is broken etc. The sum of all terms must of course equal unity.

Next we consider the second term in (6) more in detail and transform it as follows:

$$Q_2 = 3(1-P_o) \quad [P_1 + (P_o-P_1)]^2 =$$

$$= 3(1-P_o) \quad [P_1^2 + 2(P_o-P_1)P_1 + (P_o-P_1)^2] =$$

$$= 3(1-P_o) \quad [P_1^2 + 2(P_o-P_1)P_2 + 2(P_o-P_1)(P_1-P_2) +$$

$$+ (P_o-P_1)^2] \tag{7}$$

In (7) we first find the probability $3(1-P_o)P_1^2$ that one element is broken for a stress below σ_o but that the two remaining elements sustain the enhanced stress σ_1.

Secondly we find the probability $3(1-P_o)\cdot 2(P_o-P_1)\cdot P_2$ that one element is broken for a stress below σ_o, that another element is broken for a stress between σ_o and σ_1 and that the last element will sustain the stress σ_2.

We can actually derive this result directly as follows: The probability that one element is broken for a stress below σ_o and that the others will sustain this stress is $3(1-P_o)P_o^2$. The probability that one of the two remaining elements will be broken for a stress between σ_o and σ_1 given that it sustained the stress σ_o is $(P_o-P_1)/P_o$ since the fraction P_o of a large number of elements sustains the stress σ_o and the fraction (P_o-P_1) will be broken between σ_o and σ_1. Likewise the probability that the last element will sustain the stress σ_2 is P_2/P_o. Thus the probability that one particular element of the two that sustained σ_o will be broken below σ_1 and that the other will sustain σ_2 is $(P_o-P_1)P_2/P_o^2$. The particular element of these two can be chosen in $\binom{2}{1} = 2$ ways, i.e. the probability that either of the two elements that sustained σ_o will be broken below σ_1 and the remaining element will sustain σ_2 is $2(P_o-P_1)P_2/P_o^2$. Multiplying this with the probability $3(1-P_o)\ P_o^2$ obtained above yields the result $3(1-P_o)\cdot 2(P_o-P_1)P_2$ i.e. the second term in the last member of (7).

In analogy with this we can easily show that the third term in (7) represents the probability that one element

is broken below σ_o, followed by an element being broken between σ_o and σ_1 and the third element being broken between σ_1 and σ_2. Thus this term represents a case when the multiple sample in figure 1 is broken completely whereas the multiple sample holds for the cases represented by the first two terms in (7).

The last term to the right in (7) represents the case that one element is broken below σ_o followed by the two other elements being broken between σ_o and σ_1, i.e. the multiple sample is completely broken.

Next we consider the third term in equation (6) and transform it to:

$$Q_3 = 3(1-P_o)^2 \left[P_2 + (P_o-P_2)\right] \tag{8}$$

where the term $3(1-P_o)^2 P_2$ represents the case that two elements are broken below σ_o and that the remaining element sustains the stress σ_2, i.e. the multiple sample holds. The term $3(1-P_o)^2 (P_o-P_2)$ differs in that the remaining element is broken between σ_o and σ_2, i.e. the multiple sample is completely broken.

Finally the last term in the expansion in (6), namely

$$Q_3 = (1-P_o)^3 \tag{9}$$

represents the case that none of the three elements holds for the stress σ_o, i.e. the multiple sample is completely broken.

Next we define the probabilities R_k that n-k elements will be broken and k elements will sustain the enhanced stress σ_k. We thus collect the appropriate terms above and find:

$$R_3 = P_o^3 \tag{10}$$

$$R_2 = 3(1-P_o)P_1^2 \tag{11}$$

$$R_1 = 6(1-P_o)\cdot(P_o-P_1)P_2 + 3(1-P_o)^2P_2 \tag{12}$$

$$R_o = 6(1-P_o)\cdot(P_o-P_1)(P_1-P_2) + 3(1-P_o)(P_o-P_1)^2 + 3(1-P_o)^2(P_o-P_2)+(1-P_o)^2 \tag{13}$$

for the multiple sample in figure 1.

THE GENERAL SOLUTION

As mentioned before the probability that n_1 of n elements break and $n-n_1$ hold for the stress σ_o is

$$Q_{n_1} = \binom{n}{n_1} (1-P_o)^{n_1} P_o^{n-n_1}. \tag{14}$$

Given that n_1 elements are broken the stress in each element increases to

$$\sigma_{n_1} = \sigma_o \frac{n}{n-n_1} \tag{15}$$

and the probability that n_2 of the remaining $n-n_1$ elements break and $n-n_1-n_2$ hold for stresses between σ_o and σ_{n_1} is

$$Q_{n_2} = \binom{n-n_1}{n_2}(P_o-P_{n_1})^{n_2} P_{n_1}^{n-n_1-n_2} / P_o^{n-n_1} \tag{16}$$

The combined probability becomes

$$Q_{n_1,n_2} = \binom{n}{n_1}\binom{n-n_1}{n_2}(1-P_o)^{n_1}(P_o-P_{n_1})^{n_2} P_{n_1}^{n-n_1-n_2} \tag{17}$$

The stress

$$\sigma_{n_2} = \sigma_o \frac{n}{n-n_1-n_2} \tag{18}$$

and the combined probability that $n_1+n_2+n_3$ elements break and $n-n_1-n_2-n_3$ hold for stresses between σ_{n_1} and σ_{n_2} is

$$Q_{n_1,n_2,n_3} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}(1-P_o)^{n_1}(P_o-P_{n_1})^{n_2}(P_{n_1}-P_{n_2})^{n_3} P_{n_2}^{n-n_1-n_2-n_3} \tag{19}$$

Continuing in this way until

$$\sum_{i=1}^{k} n_i = n \tag{20}$$

we get

$$Q_{n_1,n_2,\ldots,n_k} = \frac{n!}{n_1!\ldots n_k!}(1-P_o)^{n_1}(P_o-P_{n_1})^{n_2}(P_{n_1}-P_{n_2})^{n_3}\ldots P_{n_{k-1}}^{n_k} \tag{21}$$

where n_k is the number of the last elements not broken.

The probability of fracture finally becomes

$$P_n = 1 - \sum Q_{n_1,n_2,\ldots,n_k} \tag{22}$$

where $n_1,n_2\ldots,n_k$ should be given all integer values >0 which satisfy the relation (20).

The number of terms N increases rapidly with the number of elements n and is found to be

$$N = 2^{n-1} \tag{23}$$

This makes it extremely hard to calculate the probability for large numbers of elements connected in parallel. In tables 1 and 2 below the expressions for 4 and 5 elements are given with the combinations of n_i to the left and the corresponding probabilities $Q_{n_1,n_2,\ldots,n_k}$ to the right.

Four elements in parallel					
n_1	n_2	n_3	n_4	Q_{n_1, n_2, n_3, n_4}	
4				P_o^4	4 hold
1	3			$4(1-P_o)P_1^3$	3 hold
2	2			$6(1-P_o)^2P_2^2$	2 hold
1	1	2		$12(1-P_o)(P_o-P_1)P_2^2$	2 hold
3	1			$4(1-P_o)^3P_3$	1 holds
2	1	1		$12(1-P_o)^2(P_o-P_2)P_3$	1 holds
1	2	1		$12(1-P_o)(P_o-P_1)^2P_3$	1 holds
1	1	1	1	$24(1-P_o)(P_o-P_1)(P_1-P_2)P_3$	1 holds

$P_o = p(\sigma_o)$, $P_1 = p(4/3\sigma_o)$, $P_2 = p(2\sigma_o)$, $P_3 = p(4\sigma_o)$

Tabell 1. Probability for not fracturing of four elements in parallel.

Five elements in parallel						
n_1	n_2	n_3	n_4	n_5	$Q_{n_1, n_2, n_3, n_4, n_5}$	
5					P_o^5	5 hold
1	4				$5(1-P_o)P_1^4$	4 hold
2	3				$10(1-P_o)^2P_2^3$	3 hold
1	1	3			$20(1-P_o)(P_o-P_1)P_2^3$	
3	2				$10(1-P_o)^3P_3^2$	2 hold
2	1	2			$30(1-P_o)^2(P_o-P_2)P_3^2$	
1	2	2			$30(1-P_o)(P_o-P_1)^2P_3^2$	
1	1	1	2		$60(1-P_o)(P_o-P_1)(P_1-P_2)P_3^2$	
4	1				$5(1-P_o)^4P_4$	1 holds
3	1	1			$20(1-P_o)^3(P_o-P_3)P_4$	
1	3	1			$20(1-P_o)(P_o-P_1)^3P_4$	
2	2	1			$30(1-P_o)^2(P_o-P_2)^2P_4$	
2	1	1	1		$60(1-P_o)^2(P_o-P_2)(P_2-P_3)P_4$	
1	2	1	1		$60(1-P_o)(P_o-P_1)^2(P_1-P_3)P_4$	
1	1	2	1		$60(1-P_o)(P_o-P_1)(P_1-P_2)^2P_4$	
1	1	1	1	1	$120(1-P_o)(P_o-P_1)(P_1-P_2)(P_2-P_3)P_4$	

$P_o=p(\sigma_o)$, $P_1=p(5/4\sigma_o)$, $P_2=p(5/3\sigma_o)$, $P_3=p(5/2\sigma_o)$, $P_4=p(5\sigma_o)$

Table 2. Probability for not fracturing of five elements in parallel.

APPROXIMATE SOLUTION

From tables 1 and 2 it is seen that the probabilities Q_{n_1,n_2}... can be grouped depending on how many elements finally sustain the load. It is now reasonable to believe that the probability that the last element will carry the load is very small. If so we can neglect the last four and eight terms respectively and in this way reduce the number of terms to one half. It may also be possible to neglect the terms where two or more elements will carry the load which reduces the terms to one half each time one element is added to the remaining elements.

When using a computer we can start from above with 5, 4, 3 ... elements left and see where adding new terms is negligible. It may be reasonable to think that when about half of the elements are broken we can neglect the probability that the rest will carry the load.

The number of terms is in this way reduced to

$$N_1 = 2^{n/2-1} \tag{24}$$

Furthermore as seen from tables 1 and 2 the terms cancelled out are the most complicated ones.

COMPARISON BETWEEN SERIAL AND PARALLEL CONNECTION

It must be remembered that what we are interested in is the probability of fracture P_n as a function of the nominal stress σ_o. Thus the numerical evaluations indicated above must be repeated for all σ_o-values for which we wish to know the probabilities of fracture. This is clearly a task to be solved with the aid of a computer.

For comparison we also consider the simple case of n elements in series for which the probability S_n for the multiple sample to be broken readily is found to be:

$$S_n = 1 - P_o^n = 1 - p(\sigma_o)^n \qquad (25)$$

As is easily understood we always have $P_n \leq S_n$ since one broken element in the series arrangement means that the multiple sample is broken whereas there is a non-vanishing probability that the parallel arrangement will hold in spite of some broken elements.

Numerical results have been worked out for a number of cases and are presented in diagrams 2 and 3 for the different values m = 3 and 5 of Weibull's parameter in (2). The dotted curves show the probability S_n of n = 1,2,3,4,5 and 10* elements in series and the full curves show the probabilities P_n for the same number of elements. It is interesting to note that for small stresses σ_o the full curves intersect and that the intersection values of σ_o increase with the m-values and are almost independent of the n-values. When $m \to \infty$ all elements have the same strength and break at $\sigma_o/\sigma_m = 1$ for both serial and parallel connection. Furthermore the spread around the average strength for the multiple samples decreases with increasing m- and n-values, both absolutely (the curves become "steeper") and relative to the average strength, which is obtained from $P(\sigma_o) = 0.5$.

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* For n=10 the strength of the four last elements in parallel is neglected ($n_k \geq 6$) which reduced the number of terms from 512 to 16.

Acknowledgement:

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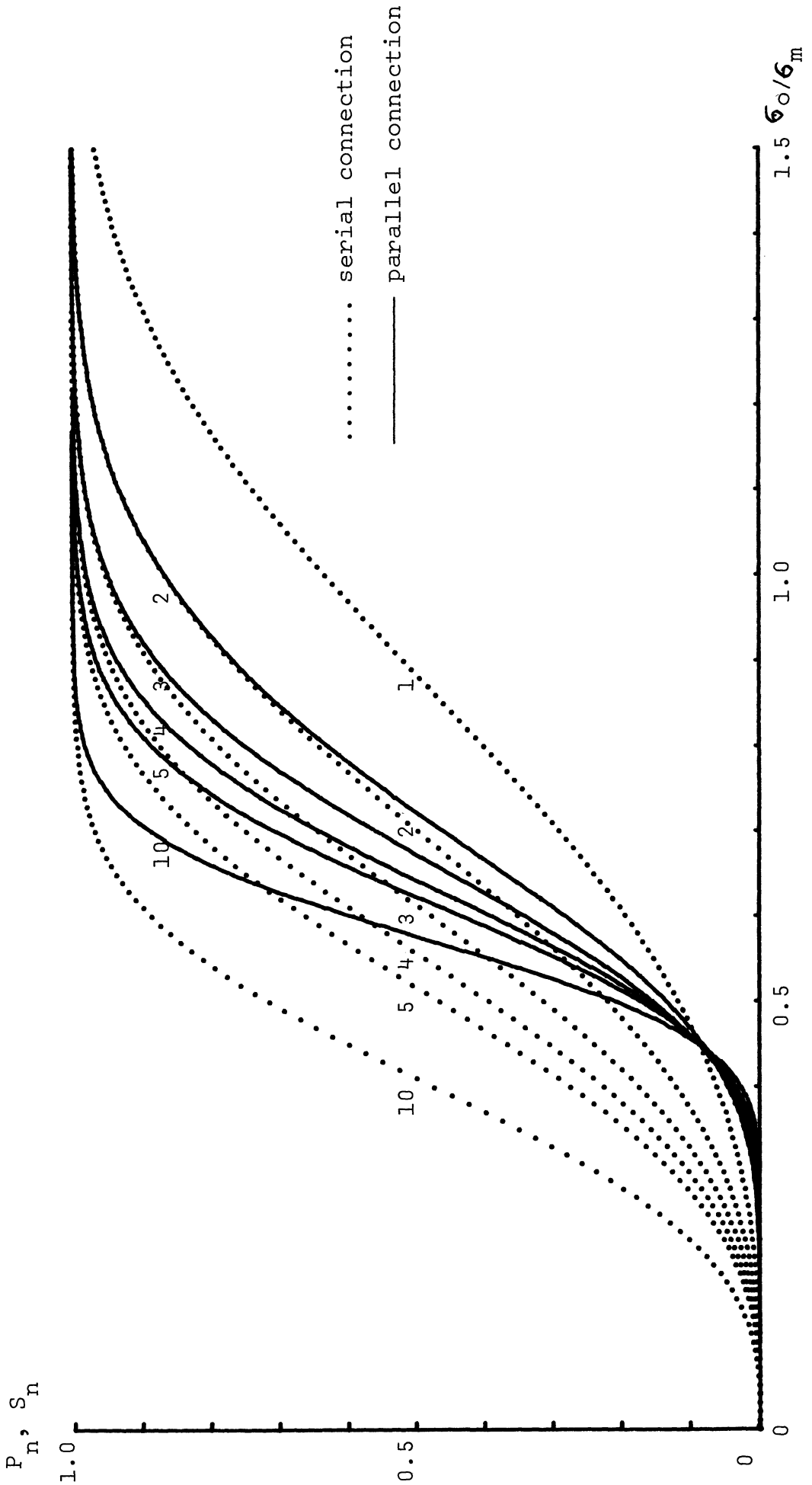

Fig. 2 Probability of fracture versus nominal stress for serial and parallel connections of one to ten elements, with the Weibull parameter m = 3

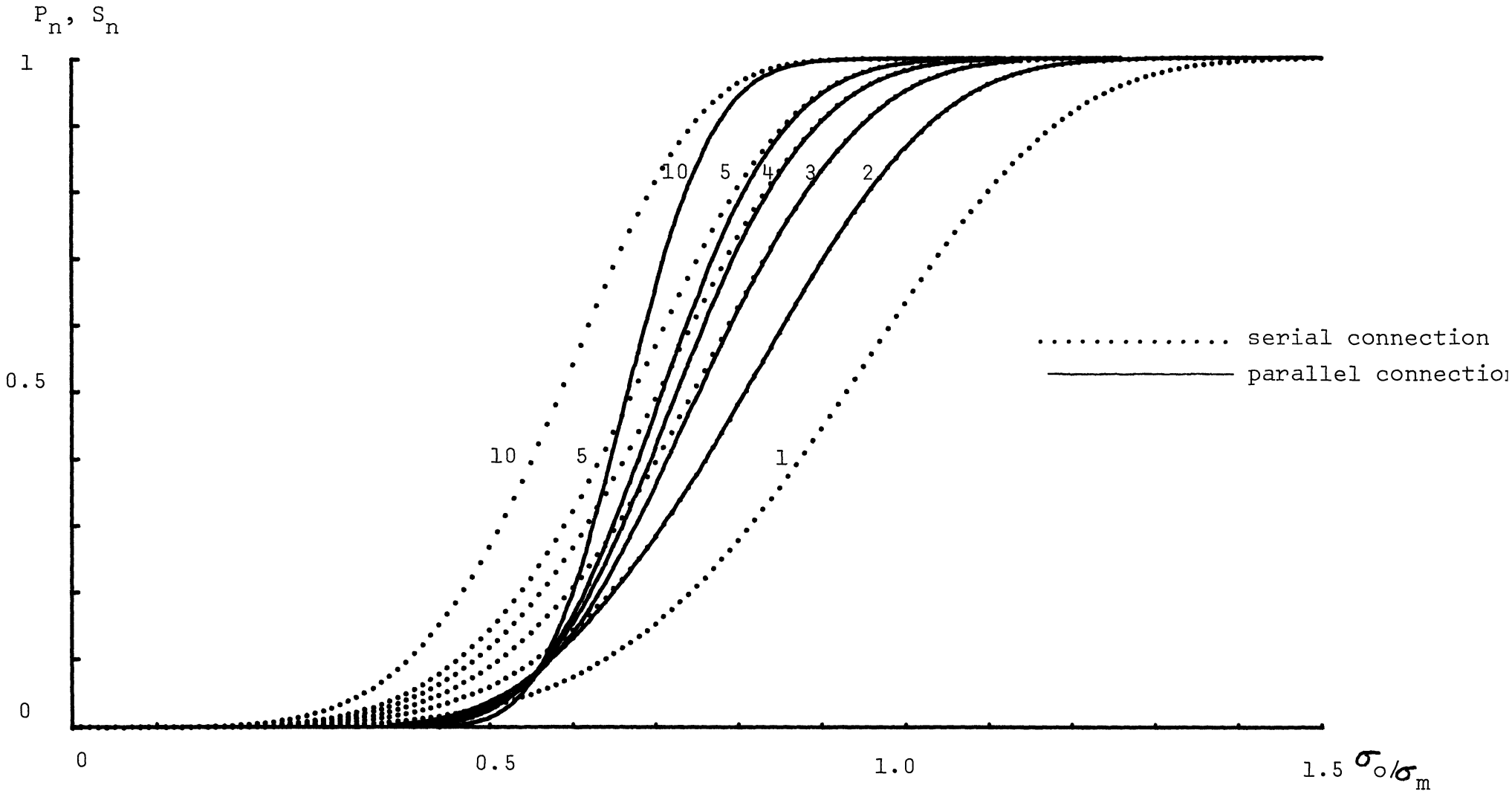

Fig. 3 Probability of fracture versus nominal stress for serial and parallel connections of one to ten elements, with the Weibull constant m = 5.