A STATISTICAL THEORY OF THE POLYAXIAL STRENGTH OF MATERIALS

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DS 1976:8 3.9.1976

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Summary

This report contains:

1. Reprint of an accepted paper presented at the second International Conference on Mechanical Behavior of Materials in Boston 1976.

2. The oral presentation of the paper.

3. References.

Introduction

The Second International Conference on Mechanical Behavior of Materials was held in Boston 16-20 August 1976.

About 600 people visited the conference excepting accompaning persons. The conference was divided in several parallell sessions as seen from the ICM-II Program Grid.

Proceedings of the Conference were delivered at the registration containing over 400 papers (2051 pages).

Invited papers were not printed in advance but should appear in a special volume.

Each lecture was limited to 20 minutes. The conference was interesting as I here came in contact with people working with other materials than rock only (concrete for example).

The paper in this report was presented the first day and showed surprisingly great interest. Five people from USA, England, Germany and Israel who worked with the same problem contacted me after the lecture for future collaboration in this area.

Session Titles

1. ELASTIC, PLASTIC AND VISCOELASTIC PROPERTIES

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ICM-II Program Grid

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SUNDAY 15 August	RECEPTION — REPUBLIC ROOM — 6:30 p.m						
MONDAY 16 August	8:30 a m	OPENING SESSION — BALLROOM — 8:30 a m - 10:00 a.m					
	9:50 a m	Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Metals and Alloys - I	Testing: Fracture and Fatigue	Creep: Microstructural Effects - I	Fatigue: Metallurgical Factors - I	Fracture: Crack Tip Phenomena - I	Structure-Property Relations: Physical Metallurgy - I
	2:00 p m.	Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Metals and Alloys - II	Testing: Dynamic Testing	Creep: Microstructural Effects - II	Fatigue: Metallurgical Factors II	Fracture: Crack Tip Phenomena II	Structure-Property Relations: Physical Metallurgy II
	3:20 p m				Fatigue: Microstructural Aspects of Crack Growth - I		

		Fairfax A	Fairfax B	Commonwealth	Constitution	Republic	Gardner	Hampton
TUESDAY 17 August	8:30 a.m	Structure-Property Relations: Physical Metallurgy - III	Testing: X-Ray Methods	Fatigue: Microstructural Aspects of Crack Growth - II	Fracture: Plasticity Analysis and Modeling - I	Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Metals and Alloys - III	Fracture: Non-Metals and Composites - I	Creep: Microsructural Effects - III
			10:10 a m Testing: Acoustic Emission	9:50 a m. Fatigue: Crack Initiation at Notches		10:10 a m Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Polymers - I		10:10 a m Creep: Interactions and Damage - I
	2:00 p.m	Structure: Temperature and Strain Rate Effects - I	Fracture: Test Methods	Fatigue: Closure, Retardation	Fracture: Plasticity Analysis and Modeling - II	Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Polymers - II	Fracture: Non-Metals and Composites - II	Creep: Interactions and Damage - II
	4:30 p m.	PLENARY LECTURE — BALLROOM - Honorable Emilio Q Daddario - Nathan E Promisel, Presiding						

		Fairfax A & B	Independence West	Independence Center	Independence East	Commonwealth	Constitution	Republic
WEDNESDAY 18 August	8:30 a m	Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Polymers - III	Fatigue: Combined Stresses	Fracture: High Strain Rates	Creep: Phenomonology and Models	Fatigue: Statistical Aspects	Fracture: Microstructural Effects I	Structure: Temperature and Strain Rate Effects - II
		9:50 a m. Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Concrete	10:30 a.m. Fatigue: Polymers and Composites - I			10:10 a.m Fatigue: Environmental Effects - I		
	2:00 p.m.	Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Composites - I	Fatigue: Polymers and Composites - II	Fracture: Environmental Effects - I	Creep: Cracking Mechanisms and Rupture	Fatigue: Environmental Effects - II	Fracture: Microstructural Effects - II	Structure: Composites and Ceramics - I
						3:20 p m. Fatigue: Crack Growth Models - I		3:20 p m Structure: Precipitation Hardening Materials
	4:30 p.m.	PLENARY LECTURE — BALLROOM - Professor Morris Cohen - Dr Edward I Salkovitz, Presiding						

		Fairfax A & B	Independence West	Independence East	Commonwealth	Constitution	Republic	Gardner
THURSDAY 19 August	8:30 a.m.	Elastic, Plastic and Viscoelastic Properties: Deformation Characterization, Composites - II	Fatigue: Design and Failure Analysis	Design: Nonmetallic Materials - I	Fatigue: Crack Growth Models - II	Fracture: Structure-Property Relations	Fracture: Environmental Effects - II	Structure: Composites and Ceramics - II
		10:30 a m Elastic, Plastic and Viscoelastic Properties: Stress Analysis, Notches and Cracks - I			9:50 a m. Fatigue: Low Cycle I		9:50 a m Fracture: Microstructural Effects - III	10:10 a m Structure: Processing
	2:00 p.m.	Elastic, Plastic and Viscoelastic Properties: Stress Analysis, Notches and Cracks - II	Fatigue: High Temperature Low Cycle - I	Design: Nonmetallic Materials - II	Fatigue: Low Cycle - II	Fracture: Design and Applications - I	Fracture: Ductile-Brittle Transition	
		3:20 p.m. Elastic, Plastic and Viscoelastic Properties: Special Topics - I		3:00 p.m. Design: Metals and Alloys - I				
	4:30 p.m.	PLENARY LECTURE — BALLROOM - Dr J H Hollomon - Dr. Walter E. Owen, Presiding						

		Fairfax A & B	Independence West	Independence East	Commonwealth	Constitution	Republic
FRIDAY 20 August	8:30 a.m.	Elastic, Plastic and Viscoelastic Properties: Special Topics - II	Fatigue: High Temperature Low Cycle - II	Design: Metals and Alloys - II	Fatigue: Surface Effects	Fracture: Design and Application - II	Fracture: Non-Metals and Composites - III
	12:00 noon	CLOSING CEREMONY — BALLROOM - Nathan E Promisel and Professor V. Weiss					

PROCEEDINGS

of the

SECOND INTERNATIONAL CONFERENCE

on

MECHANICAL BEHAVIOR OF MATERIALS

16-20 August 1976

Sheraton-Boston Hotel
Boston, Massachusetts

Under the Auspices of

Federation of Materials Societies

1098

A STATISTICAL THEORY OF THE POLY-AXIAL STRENGTH OF MATERIALS

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INTRODUCTION

Most conventional strength criteria assume that there is no scatter in the strength. This means that the local strength is the same for the whole material, although the material may have different strength values in different directions.

Everyone who has to deal with measuring strength knows, however, that a scatter exists. Weibull for example pointed out that this leads to a strength volume dependence.

The present paper shows that this also leads to a variation in strength with the intermediate principal stress, that agrees with experimental results.

In this work it is assumed that a Weibull strength distribution exists described by the equation

$$P=1-\exp(-k\tau^{M})$$

where P is the probability of rupture, and τ the shearing stress.

This distribution is very similar to the ln-normal distribution if 1/M is the standard deviation. The Weibull distribution, however, is more easy to use from a mathematical point of view.

A QUALITATIVE COMPARISON WITH THE MOHR-COULOMB CRITERION

Let us assume that line (a) in Fig.1 represents the Mohr-Coulomb fracture criterion. If there is no scatter in the strength the fracture will take place at definite values of σ_1 and σ_3, and at a definite angle β, that is twice the fracture angle between the fracture plane and the minor prin-

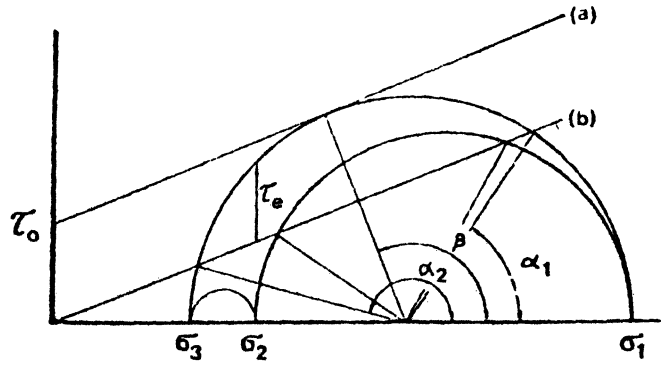

Fig 1. Mohr's circles with fracture criterion.

cipal stress σ_3. From the figure we can also see that the intermediate principal stress σ_2 has no effect on the strength.

In reality, there is a statistical distribution of the strength above and below line (a), and as a consequence a distribution of the shear strength over different angles around the classical angle (β). To deal with this, let us move line (a) to the position (b). The line now represents the friction between two fractured surfaces, and we may see from the figure that there are angles where fracture is impossible ($\alpha>\alpha_2$) and ($\alpha<\alpha_1$) because the friction here exceeds the shearing stress. For $\alpha_1<\alpha<\alpha_2$, however, there is a probability for fracture. This probability is a function of the size of the angle and the effective shearing stress τ_e. We can also see that σ_2 has an effect on the strength as one of the smaller Mohr circles exceeds the line (b).

STRESSES IN THE POLYAXIAL STRESS FIELD

In the principal stress space it is easy to find the normal stress σ_n and shearing stress τ_n in a plane, the normal of which (OP Fig.2) has a given direction. Let us define the effective shearing stress as

$$\tau_e=|\tau_n|-\mu\sigma_n \quad \ldots\ldots\ldots\

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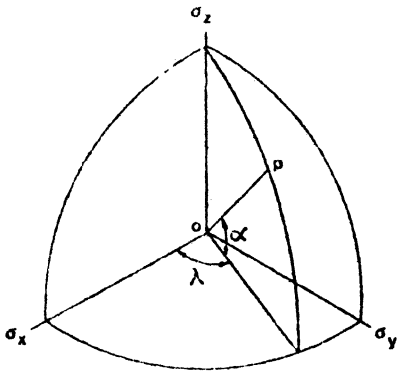


Fig 2. Unit sphere with principal stress axis.

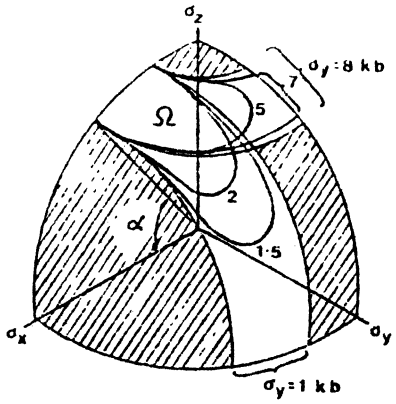


Fig 3. Computed solid angles for fracture at various stress states.

σ_x and σ_z. It is also evident that the solid angle where fracture can occur at first decreases with increasing σ_y, reaches a minimum, and then again increases. This indicates that the strength has a maximum when the intermediate principal stress lies approximately half-way between the other two.

CALCULATION OF THE STRENGTH

By using the Weibull distribution, the probability of rupture may be expressed in the form:

$$P=1-\exp(-X) \quad \ldots\ldots(2)$$

$$X=k\int \tau_e^{M} d\Omega(\tau_e>0) \quad \ldots\ldots(3)$$

where k and M are material parameters.

Experimental strength values from a uniaxial compression test corresponding to 50 per cent rupture, give us the X and k-values and we can then compute the strength at any combination of the principal stresses with constant X and k values. Values of μ and M are first determined from other simple experiments.

The strength is presented in Fig.4 on the σ_x, σ_y plane with $\sigma_z=0$ and $\mu=1$ for the cases of M=2, 3, 5, 100, and infinity. We note that the influence of the intermediate principal stress decreases with increasing M, and when M tends to infinity i.e. no scatter in the strength, the results are in agreement with the classical Mohr-Coulomb theory where σ_2 has no influence on the strength. For M=2 and $\mu=0$, the result will

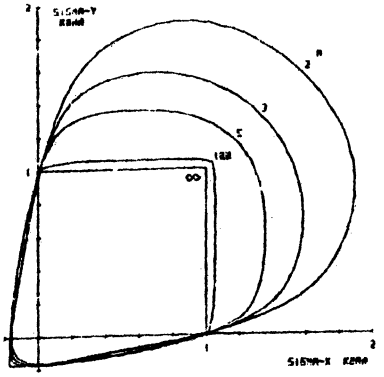


Fig 4. Strength of materials for μ=1, and different M when σ_2=0.

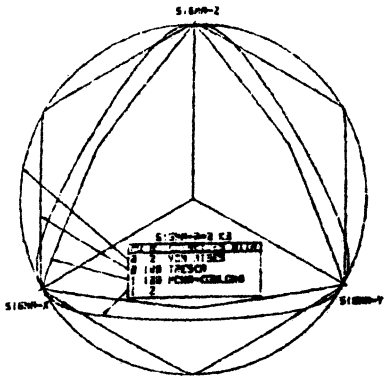


Fig 5. Stress relations at fracture for the diviatoric plane.

coincide with the von Mises theory, and the strength curves become ellipses.

COMPARISON WITH CONVENTIONAL THEORIES

A further comparison between different strength criteria is shown in Fig. 5 for the deviatoric plane in the principal stress space where the strength is calculated for the mean stress σ_o equal to 2 kb, and for different μ and M values. As in Fig.4 M=100 gives results which are almost equal to those for M=equal to infinity. For special combinations of μ and M the statistical theory coincides with earlier strength criteria. These are: Von Mises μ=0 and M=2, Tresca μ=0 and M=∞, and Mohr-Coulomb an arbitrary μ and M=∞.

RELATIONS BETWEEN STRENGTH, μ AND M

A useful relation between μ and M can be determined by using the uniaxial (C_1) and biaxial compressive strength (C_2) i.e.

$$C_2/C_1=(\sqrt{1+\mu^2}+\mu)^{1/M} \quad \ldots\ldots(4)$$

A relationship between (C_1) and the tensile strength (K) is also found and gives

$$C_1/K=(\sqrt{1+\mu^2}+\mu)^2+\mu(\sqrt{1+\mu^2}-1)(\sqrt{1+\mu^2}+\mu)\cdot M^{-1\cdot 8} \quad \ldots\ldots(5)$$

This formula is exact for M=1 and M=infinity and and is correct within a few per cent for other M-values.

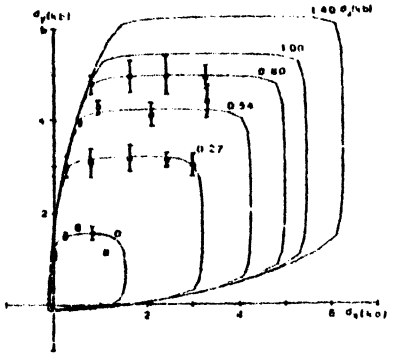


Fig 6. Strength of sandstone with σ_z as a parameter.

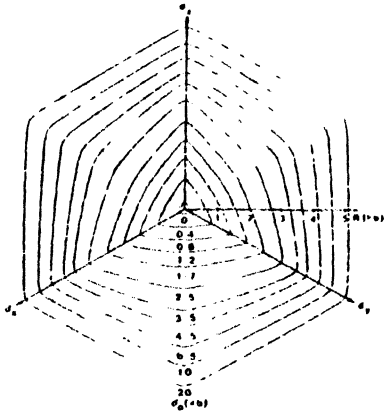


Fig 7. Strength of sandstone with σ_o as a parameter.

The μ-value may also be a function of σ_n. For example it has been shown empirically that for many rock materials

$$\mu=\mu_o/(1+\mu_o\sigma_n/\tau_x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where μ_o is the μ-value for $\sigma_n=0$ and τ_x the friction stress when σ_n tends to infinity.

COMPARISON BETWEEN EXPERIMENTAL AND CALCULATED STRENGTH

Experimental and calculated values of the strength give good agreement for several rock materials. For example, for sandstone $\mu=1.4/(1+0.37\sigma_n)$, M=7, and $C_1=1.2$ kb. The strength is shown in Fig.6 and 7 with σ_z and σ_o as parameters. As seen from Fig.6 the agreement is very good between experimental and calculated strength.

SUMMARY

The theory gives a general model of the strength of a material, where the independent parameters of the theory are the internal friction and the variance in the strength distribution. The theory shows good agreement with experimental results and describes well the variation of the strength with the intermediate principal stress. Some well known strength criteria are found to be special cases of the present criterion. One feature of the theory is that is gives a smooth yield surface without the sharp corners inherent in the Mohr-Coulomb and Tresca criteria.

Oral presentation at the Sec. Int. Conf. on Mech. Behavior of Metals. Boston 1976.

A STATISTICAL THEORY OF THE POLYAXIAL STRENGTH OF MATERIALS

Introduction

This presentation will deal with a new theory we have used for rock material. The theory is however applicable for all materials, so I hope you may find some interest in it. The correct word in this case "Triaxial" has already been occupied for the case when two of the principal stresses are equal. So in this case when all three principal stresses are different we have to chose a new word "Polyaxial or Multiaxial".

Most of the conventional strength criteria stated that the intermediate principal stress has no influence on the strength.

On the other hand experimental work shows an effect of this principal stress, expecially for brittle materials such as rock. Many authors have pointed out the lack of a theory that describes this effect. Von Mises' theory shows such an effect but the agreement with experimental results is not very good, excepting ductile materials. While working with this problem I got the idea of the statistical theory.

Statistics - A Matter of Probability

Conventional strength criteria also assume that there is no scatter in the strength. This means that the local strength is the same for the whole material, and consequently all test pieces from a material have exactly the same strength.

Everyone who has to deal with measuring strength knows, however, that a scatter exists.

Fig. 1 shows an example of this, and you can see here that some pieces break at low stresses and some are stronger and need very high stresses for fracture. If these pieces are connected in series, as in a chain, the strength of the chain will be equal to the strength of its weakest link. This is popularly called the "weakest link theory".

By adding new links and making the chain longer an even weaker link may be introduced and the chain will be weaker. There is however no possibility to make the chain stronger by adding new links. Let us now use this idea on a solid material body replacing the links by fracture angles.

Fig. 2 shows the Mohr-Coulomb fracture criterion (line a) and the three Mohr circles. If there is no scatter in the strength the fracture will take place at definite values of σ_1 and σ_3, and at a definite angle β, that is twice the fracture angle between the fracture plane and the minor principal stress σ_3. From the figure we can also see that the intermediate principal stress has no effect on the strength, as none of the smaller circles reach the fracture curve (a).

Let us now assume that the material has a strength distribution in different directions in the same way as the links with different strength are distributed in a chain.

To deal with this, let us move line (a) to the position (b). The line now represents the internal friction or the friction between two fractured surfaces, and we may see from the figure that there are angles where fracture is impossible ($\alpha < \alpha_1$ and $\alpha > \alpha_2$ because the friction here exceeds the shearing stress. For $\alpha_1 < \alpha < \alpha_2$, however, there is a probability for fracture. This probability is a function of the effective shearing stress τ_e. We can also see that σ_2 has an effect on the strength as one of the smaller Mohr circles exceeds the line (b).

For an exact solution we must go into the tree dimensional stress space.

Fig. 3 shows a unit sphere with the tree principal stress axes. In this case it is easy to find the normal stress σ_n and shearing stress τ_n in a plane, the normal of which (OP) has a given direction. It is now possible to calculate the solid angle (Ω) where fracture is possible.

Fig. 4 shows the results for Ω corresponding to the internal friction coefficient $\mu = 1$, $\sigma_x = 8$ kb, $\sigma_z = 1$ kb, and with σ_y as a parameter. In the shaded area where τ_e is always negative

no rupture can take place for any value of σ_y between σ_x and σ_z. It is also evident that the solid angle where fracture can occur at first decreases with increasing σ_y reaches a minimum, and then again increases. This is equivalent with shortening and lengthening the chain mentioned before, and indicates that the strength has a maximum when the intermediate principal stress lies approximately half-way between the other two.

Calculation of the Strength

By using for example the Weibull distribution in Fig. 1, the probability of rupture may be computed for different combinations of the principal stresses.

Experimental strength values from a uniaxial compression test corresponding to 50 per cent rupture, give us the total probability in all directions and we can then compute the strength at any combination of the principal stresses for this constant value of the probability. Values of μ and M are first determined from other simple experiments.

The strength is presented in Fig. 5 on the σ_x, σ_y plane with $\sigma_z = 0$ and $\mu = 1$ for the case of M = 2, 3, 5, 100, and infinity. We note that the influence of the intermediate principal stress decreases with increasing M, and when M tends to infinity i.e. no scatter in the strength, the results are in agreement with the classical Mohr-Coulomb theory where σ_2 has no influence on the strength. For M = 2 and $\mu = 0$, the result will coincide with the von Mises theory, and the strength curves become ellipses.

Comparison with Conventional Theories

A further comparison between different strength criteria is shown in Fig. 6 for the deviatoric plane in the principal stress space where the strength is calculated for the mean stress σ_o equal to 2 kb, and for different μ and M values. For special combinations of μ and M the statistical theory coincides with earlier strength criteria.

These are; Von Mies $\mu = 0$ and $M = 2$, Tresca $\mu = 0$ and $M = \infty$, and Mohr-Coulomb and arbitrary μ and $M = \infty$. For other more reasonable values of μ and M the curves are smooth without sharp corners.

Relations Between Strength, μ and M

A simple relation between biaxial - and uniaxial strength is found that gives

$$\frac{C_2}{C_1} = (\sqrt{1+\mu^2}+\mu)^{1/M}$$

Fig. 7 shows the relation, that for reasonable values of μ and M varies between 1 and 6.

Another useful relation is the ratio between compressive and tensile strength that gives

$$\frac{C}{K} = (\sqrt{1+\mu^2}+\mu)^2 + (\sqrt{1+\mu^2}+\mu)(\sqrt{1+\mu^2}-1)\cdot\frac{\mu}{M^{1.8}}$$

This formula is exact for $M = 1$ and infinity and is correct within a few per cent for other M-values. For $M = \infty$ the relation reduces to the first term.

Fig. 8 shows this relation with a variation between 1 and 80. It may be noted that Griffith here stated a ratio of 8, which gives a μ-value slightly greater than 1 (a reasonably value).

Fig. 9 shows the shear strength of granite and mild steel. This figure shows that the fracture curve can not always be represented by a stright line and the internal friction coefficient becomes a function of the normal stress.

I have used a relation

$$\mu = \mu_0/(1+\mu_0\sigma_n/\tau_x)$$

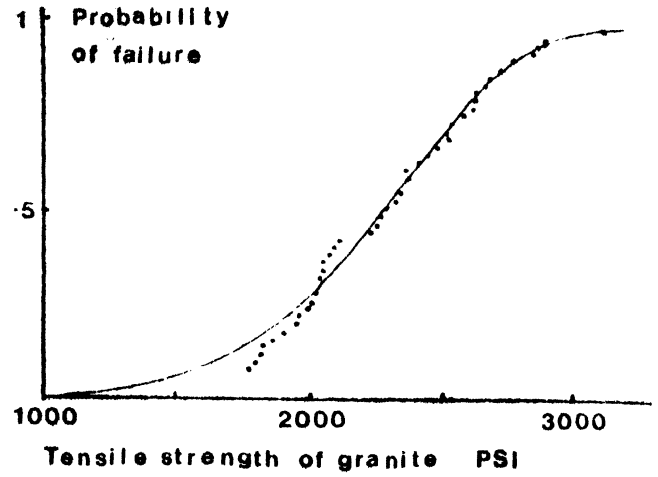
which is based upon experimental results. By using a computer this expression may easily be used instead of a constant μ-value.

Comparison Between Experimental and Calculated Strength

Experimental and calculated strength values show good agreement for several rock materials. Fig. 10 to 15 show the results for syenite, sandstone, and dolomite.

Summary

The theory gives a general model of the strength of a material, where the independent parameters of the theory are the internal friction and the variance in the strength distribution. The theory shows good agreement with experimental results and describes well the variation of the strength with the intermediate principal stress. Some well known strength criteria are found to be special cases of the present criterion. One feature of the theory is that is gives a smooth yield surface without the sharp corners inherent in the Mohr-Coulomb and Tresca criteria.



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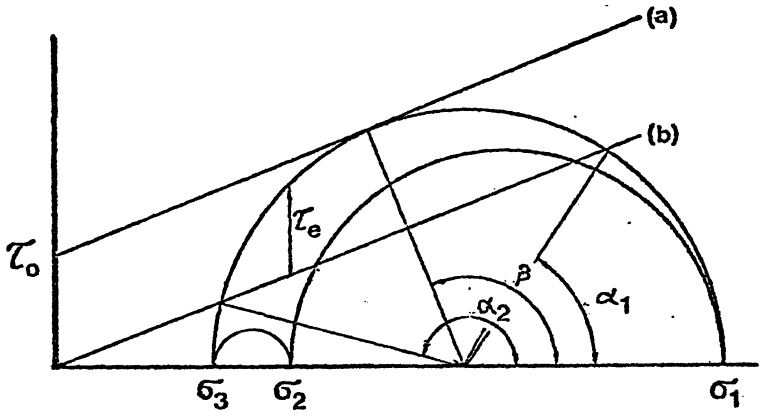
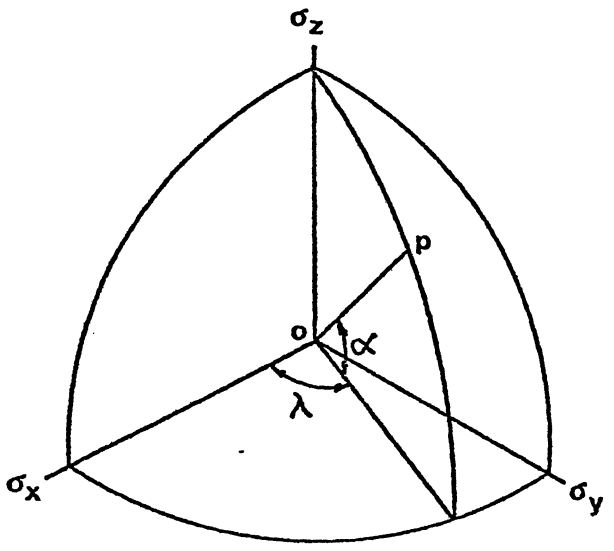


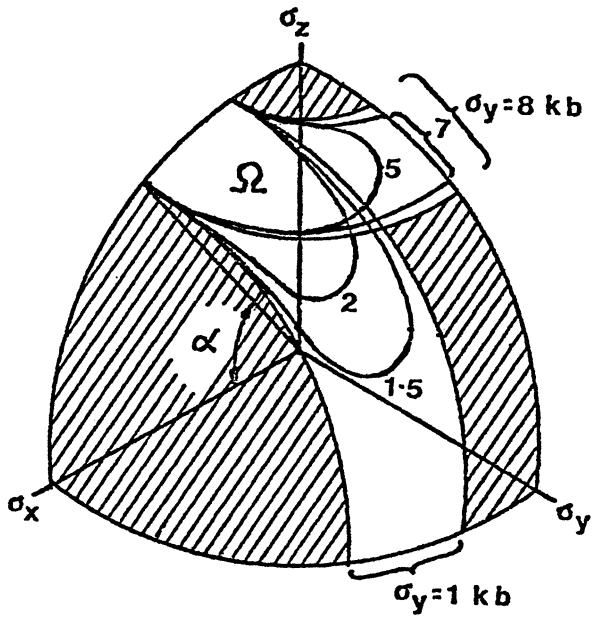
2

Mohr's circles with fracture criterion



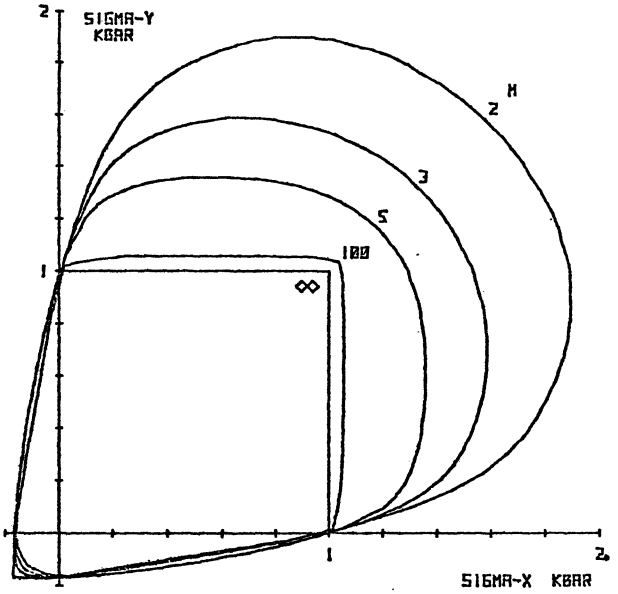
3

Unit sphere with principal stress axis



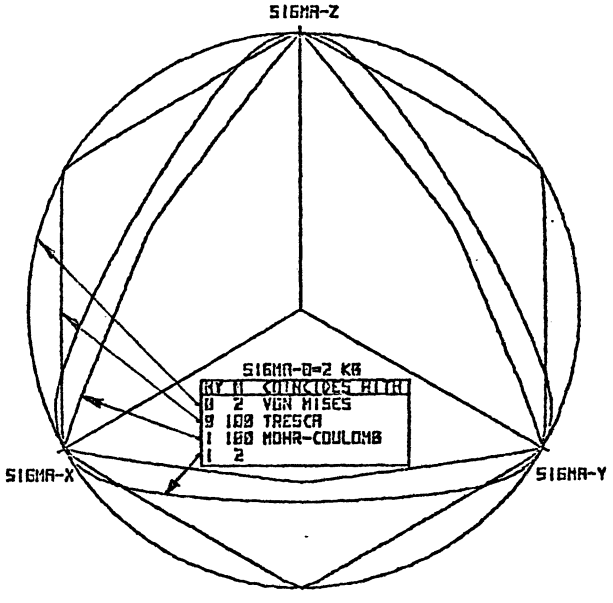
4

Computed solid angles for fracture at various stress state



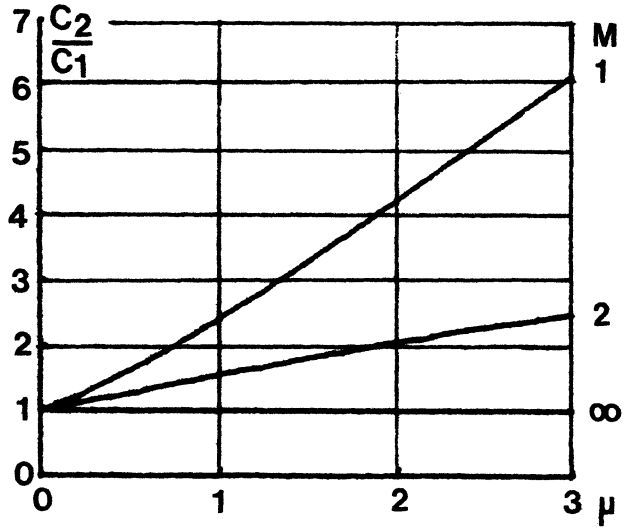
5

Strength of materials for u=½ and different M when σ_z=0



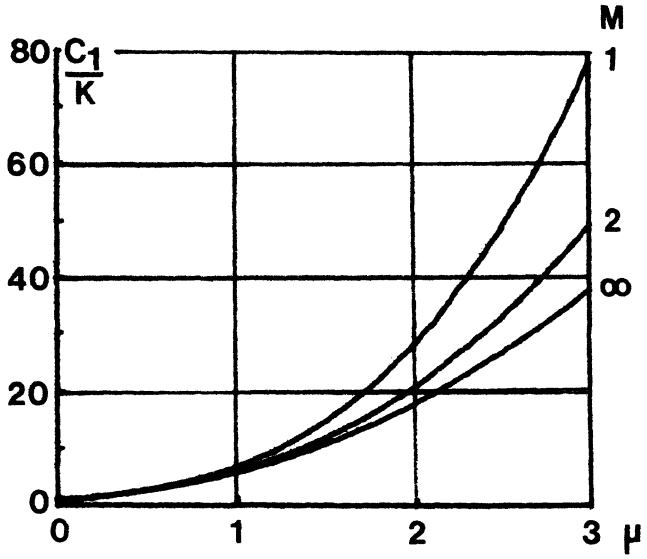
6

Stress relations at fracture for the deviatoric plane



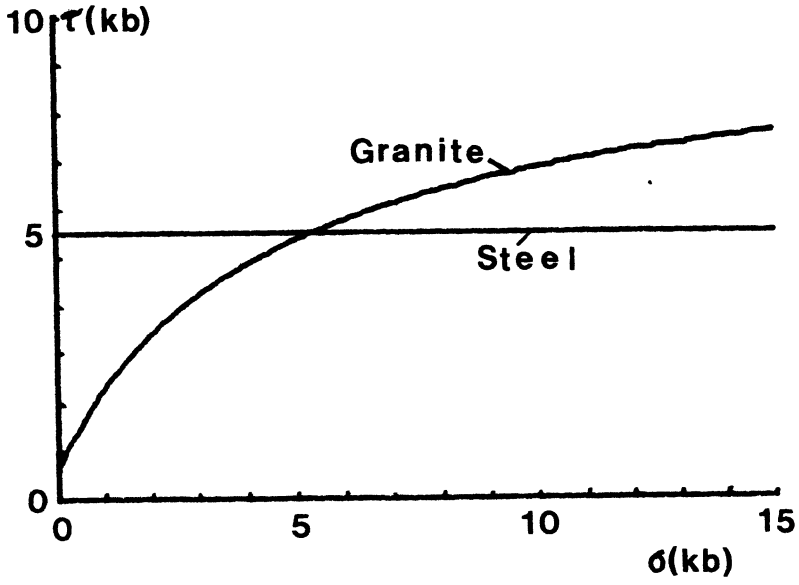
7

Relation between biaxial and uniaxial compressive strength



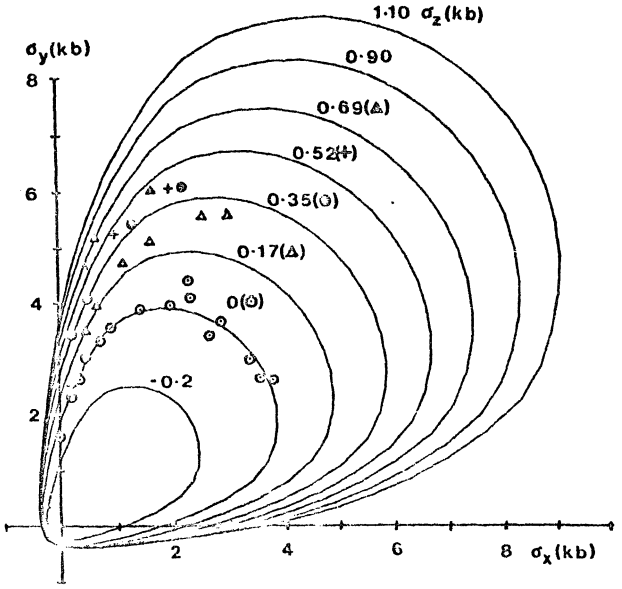
8

Relation between compressive and tensile strength



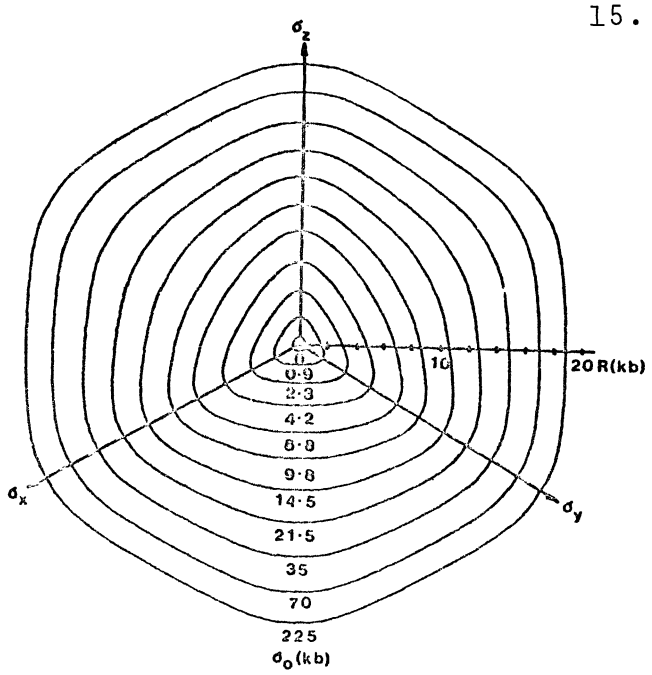
9

Shear strength versus normal stress



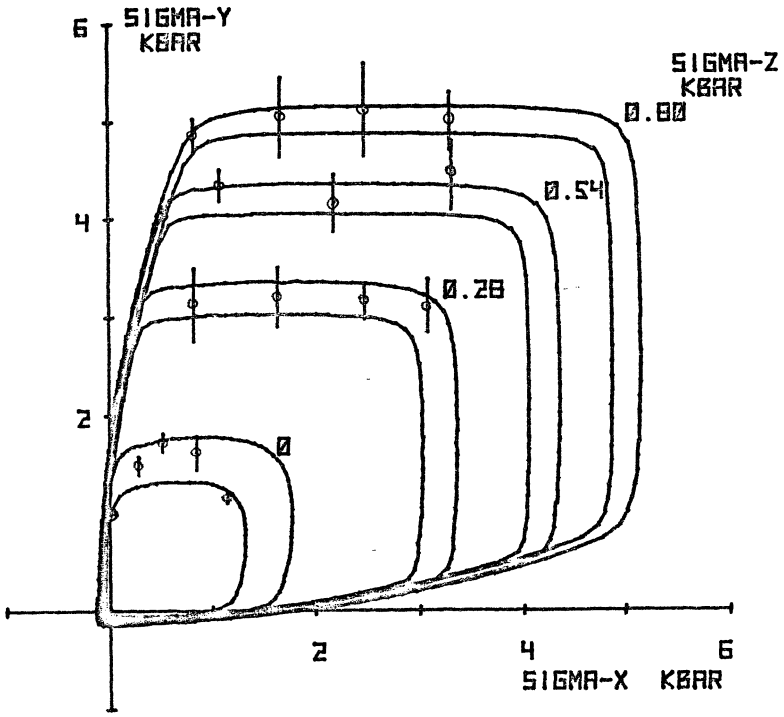
Calculated and experimental strength of syenite

10



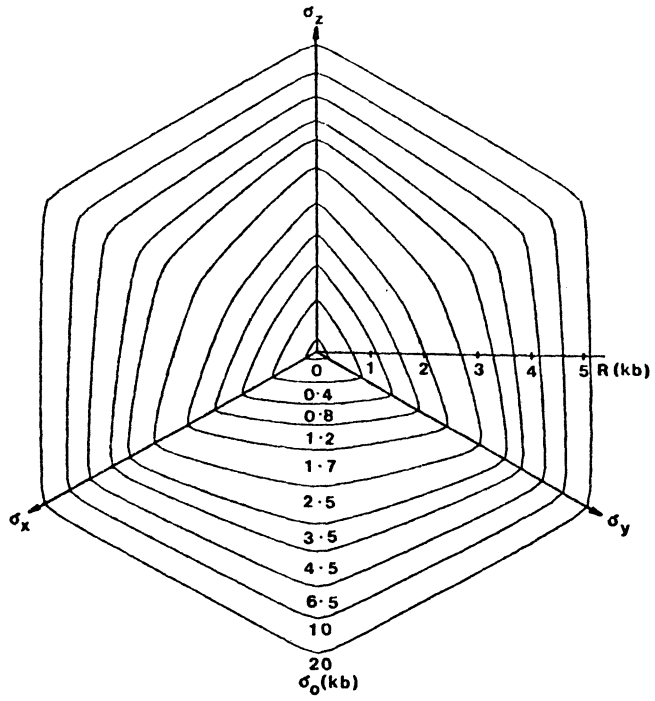
Calculated strength of syenite for the deviatoric plane at different mean stresses σ_o.

11



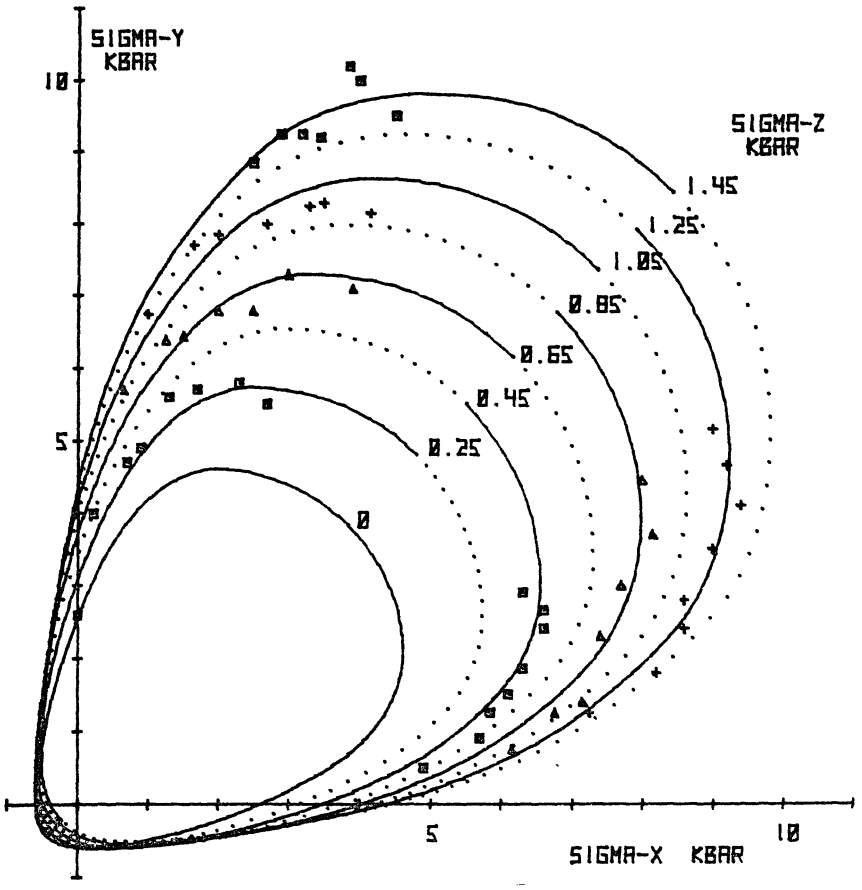
CALCULATED AND EXPERIMENTAL STRENGTH OF SANDSTONE WITH STANDARD DEVIATIONS.

12



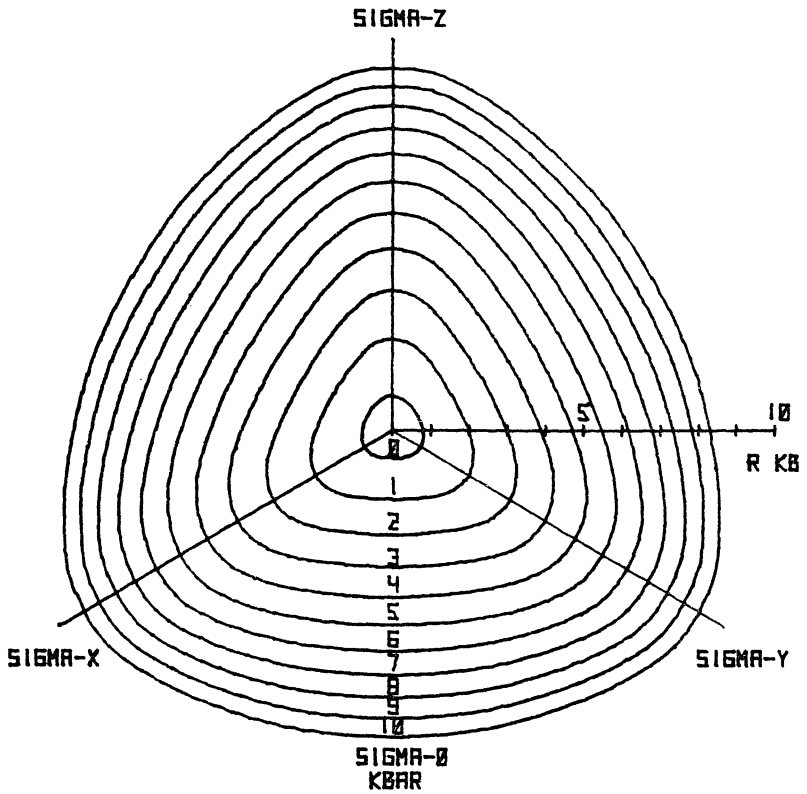
Calculated strength of sandstone for the deviatoric plane at different mean stresses σ_o.

13



14

STRENGTH OF DUNHAM DOLOMITE WITH SIGMA-Z AS A PARAMETER



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STRENGTH OF DUNHAM DOLOMITE WITH SIGMA-0 AS A PARAMETER

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