

A STATISTICAL THEORY OF THE POLYAXIAL
STRENGTH OF MATERIALS

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A STATISTICAL THEORY OF THE POLYAXIAL STRENGTH OF MATERIALS SUMMARY

A complete model of the strength in the principal stress space has been derived from a probabilistic model of the strength distribution. Some well known strength criteria are found to be special cases of the present criterion. The independent parameters of the theory are the internal friction coefficient and the variance of the strength distribution. Both of these are derivable from simple experiments. The theory shows good agreement with experimental results. One feature of the theory is that it gives a smooth yield surface without the sharp corners inherent for example in the Mohr-Coulomb and Tresca criteria.

UNE THEORIE STATISTIQUE DE LA RESISTANCE POLYAXIALE RESUME

Un modèle complet de la résistance dans la surface de contrainte principale a été dérivé d'un modèle de la probabilité de la distribution de résistance. On a trouvé que quelques critères de résistance bien connus sont des cas particuliers du critérium présent. Les paramètres indépendants de la théorie sont le coefficient de la frottement interne et la variance de la distribution de résistance. Tous les deux ont été dérivés des expériences simples. La théorie est en accord avec les résultats expérimentaux. Une caractéristique de cette théorie est qu'elle permet une surface lisse d'élasticité sans les coins aigus qui sont compris dans les critères de Mohr-Coulomb et de Tresca.

EINE STATISTISCHE THEORIE DER MEHRACHSIGEN FESTIGKEIT ZUSAMMENFASSUNG

Ein vollständiges Modell der Festigkeit im Hauptspannungsraum ist aus einem Wahrscheinlichkeitsmodell der Festigkeitsverteilung hergeleitet worden. Es hat sich gezeigt, dass einige bekannte Festigkeitskriterien Spezialfälle des Kriteriums der vorgelegten Theorie sind. Die unabhängigen Parameter der Theorie sind der innere Reibungskoeffizient und die Varianz der Festigkeitsverteilung. Diese beiden sind aus einfachen Experimenten ableitbar. Die Theorie steht in guter Übereinstimmung mit experimentellen Resultaten. Aus dieser Theorie resultiert u.a. eine glatte Festigkeitsfläche ohne die scharfen Ecken, die zu den Mohr-Coulomb und Tresca-Kriterien gehören.

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INTRODUCTION

The basic concept of the present theory has been described previously by assuming a statistical distribution of the strength of a material and the appearance of an internal friction depending of the normal stress (Lundborg 1972).

When a material has the same strength in all directions and there is no internal friction, fracture occurs when the shearing stress in any direction reaches a critical value, the shearing strength of the material. This is the well known Tresca criterion.

When there is no internal friction, but the local strength is statistically distributed, the resulting strength of the material is given by the integral of the probability for fracture over the entire volume of the material. This reasoning forms the basis for the Weibull statistical theory characterizing the strength of materials.

When a material exhibits both internal friction and a statistically distributed strength, the resulting strength of the material must be found by integrating the probability for fracture over all local shearing planes where the effective shearing stress is positive. This is the basic concept used in the present theory where the probability for fracture has been integrated over the solid angle in the stress space where the effective shearing stress is positive.

Because of its ease in representation the Weibull distribution has been used in this treatment, but naturally it is possible to use other probabilistic models, such as the normal distribution.

STRESSES IN THE POLYAXIAL STRESS FIELD

In the polyaxial stress state we can calculate the stresses in a plane, whose normal (OP Fig. 1) has the direction cosines l , m and n to the σ_x , σ_y and σ_z axes, respectively. These are:

$$\sigma_n = l^2 \sigma_x + m^2 \sigma_y + n^2 \sigma_z \quad \dots\dots\dots(1)$$

and

$$\tau_n^2 = l^2 \sigma_x^2 + m^2 \sigma_y^2 + n^2 \sigma_z^2 - \sigma_n^2 \quad \dots\dots\dots(2)$$

where

$$l^2 + m^2 + n^2 = 1. \quad \dots\dots\dots(3)$$

From Fig. 1 we get the relations

$$\left. \begin{aligned} l &= \cos \alpha \cos \lambda \\ m &= \cos \alpha \sin \lambda \\ n &= \sin \alpha \end{aligned} \right\} \dots\dots\dots (4)$$

Let us now define the effective shearing stress as

$$\tau_e = |\tau_n| - \mu \sigma_n \dots\dots\dots (5)$$

where $\mu \sigma_n$ is the internal friction stress.

By using equations (1)-(5) we can calculate the solid angle (Ω) where $\tau_e > 0$, which are the surface areas shown in Fig. 2 divided by the unit radius.

The results for Ω shown in Fig. 2 correspond to $\sigma_x = 8 \text{ kb}$, $\sigma_z = 1 \text{ kb}$, $\mu = 1$, and σ_y as a parameter. In the shaded angle where $\tau_e < 0$ no rupture

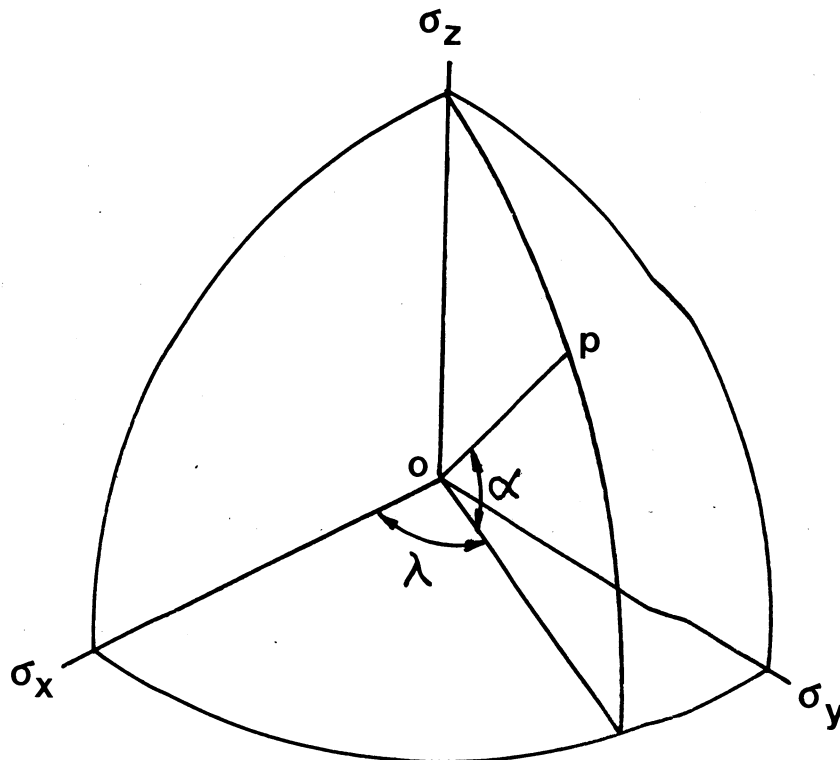


Fig. 1. Unit sphere with principal stress axes.

can take place for any value of σ_y between σ_x and σ_z . It is also evident that the solid angle where fracture can occur at first decreases with increasing σ_y , reaches a minimum, and then again increases when $\sigma_z < \sigma_y < \sigma_x$. This fact indicates that the strength has a maximum when the intermediate principal stress lies approximately half-way between the other two.

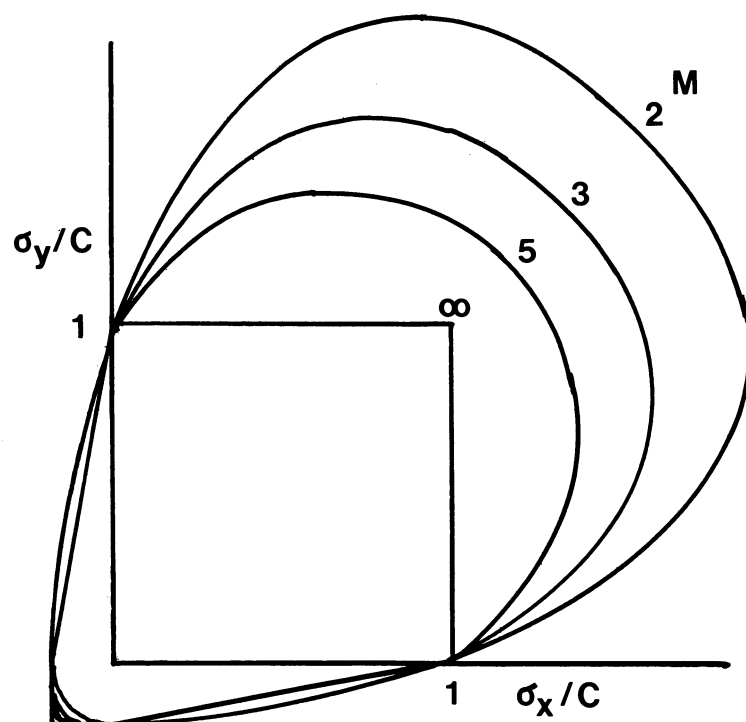


Fig. 3. Strength variation with σ_x at different M when $\mu = 1$ and $\sigma_z = 0$.

A comparison between different strength criteria is shown in Fig. 4 for the deviatoric plane in the principal stress space where the statistical theory is calculated for the mean stress σ_0 equal to zero, a uniaxial strength of 2 kb, $\mu = 1$, and $M = 2$.

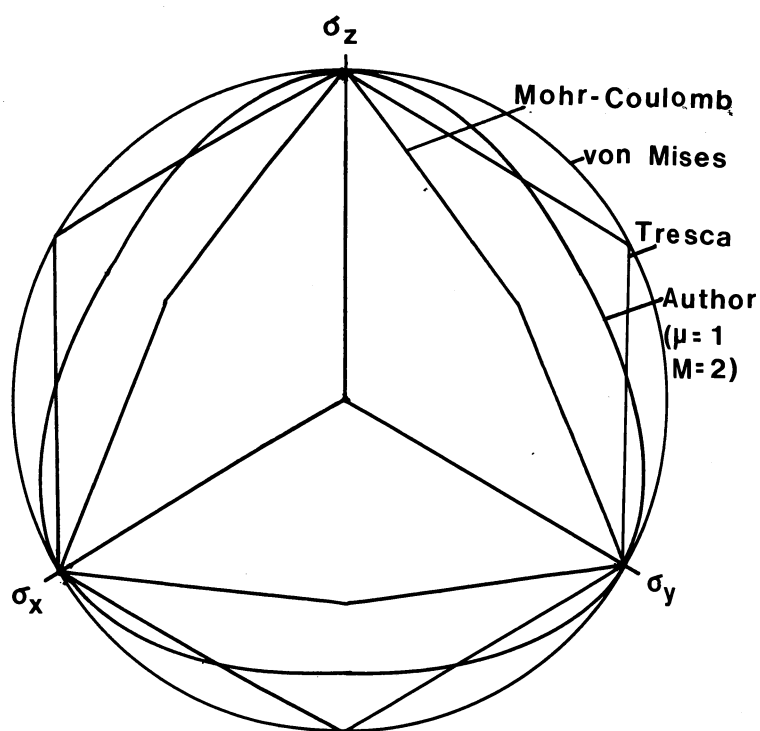


Fig. 4. Comparison between different strength theories.

For special combinations of μ and M the statistical theory coincides with earlier strength criteria as shown in Table 1.

Table 1. Values of μ and M for which the statistical theory coincides with other criteria.

Strength criteria	μ	M
von Mises	0	2
Tresca	0	∞
Mohr-Coulomb	arbitrary	∞

RELATION BETWEEN STRENGTH, μ AND M

Determination of a relation between μ and M can be made using the uniaxial and biaxial compressive strength. If we put $\sigma_z = \lambda = 0$, we get from equations (1)-(5).

$$\sigma_n = 1^2 \sigma_x \dots\dots\dots (8)$$

$$\tau_n = 1 \cdot n \sigma_x \dots\dots\dots (9)$$

and

$$\tau_e = 1(n-\mu) \sigma_x \dots\dots\dots (10)$$

From Fig. 1 or 2 it may be seen that this state of stress is held constant when rotating OP around either the σ_x ($\sigma_y = \sigma_z$) or σ_z ($\sigma_y = \sigma_x$) axis.

Using equation (4) in (10) this gives for $\lambda = 0$

$$\tau_e = \sigma_x \cos \alpha (\sin \alpha - \mu \cos \alpha) \dots\dots\dots (11)$$

When X is held constant as stated above, equations (7) and (11) give

$$\left(\frac{\sigma_{20}}{\sigma_{10}}\right)^M = \frac{\int_0^{\pi/2} [\cos \alpha (\sin \alpha - \mu \cos \alpha)]^M \sin \alpha \, d\alpha}{\int_0^{\pi/2} [\cos \alpha (\sin \alpha - \mu \cos \alpha)]^M \cos \alpha \, d\alpha} \dots\dots\dots (12)$$

where σ_{10} and σ_{20} are the uniaxial and biaxial compressive strengths, respectively. For $\alpha = \alpha_0$ τ_e is zero and (11) gives

$$\mu = \tan \alpha_0 \dots\dots\dots (13)$$

it can be shown (see Appendix) that equation (12) gives

$$\left(\frac{\sigma_{20}}{\sigma_{10}}\right)^M = \sqrt{1 + \mu^2} + \mu \dots\dots\dots (14)$$

This is a useful relationship between the Weibull parameter, friction coefficient and strength ratio.

Note in this section that α is the α -value for $\lambda = 0$.

THE μ -VALUE AS A FUNCTION OF THE NORMAL STRESS

In the literature the μ -value is usual considered independent of the normal stress. However, when the friction stress is expressed with the simple relation $\tau_f = \mu(\sigma_n) \sigma_n$, μ is not a constant over a wide range of normal stresses. It is noted that τ_f increases slowly with σ_n for high values of σ_n . In this region many expressions are given that all show a decreasing μ -value with increasing σ_n . Lundborg (1968) has used the relation

$$\mu = \mu_0 / (1 + \mu_0 \sigma_n / \tau_x) \quad \dots\dots\dots (15)$$

which is based on experimental results. μ_0 is a constant and τ_x the friction stress when σ_n tends to infinity. By using a computer, the expression (15) may easily be used instead of a constant μ -value.

COMBINED TENSILE AND COMPRESSIVE STRESSES

It is possible to extend the analysis to combined tensile and compressive stresses, in a number of different ways.

In this paper, it is assumed that the friction will still decrease when $\sigma_n < 0$ so that the calculation will hold throughout the region. Other alternatives may be to let the friction be zero for $\sigma_n < 0$, or according to Griffith (1924) let the calculation hold only when $\sigma_1 + 3\sigma_3 > 0$ and then let σ_3 be constant (σ_1 and σ_3 are the major and minor principal stress respectively).

COMPARISON BETWEEN EXPERIMENTAL AND CALCULATED STRENGTH

Experimental results from Hoskins (1969) on trachyte and from Akai and Mori (1967) on sandstone have given

$$\left. \begin{aligned} \mu &= 0.9 / (1 + \frac{0.9 \sigma_n}{12}) \\ M &= 1.2 \\ X &= 0.2702 \end{aligned} \right\} \dots\dots\dots (16)$$

for trachyte and

$$\left. \begin{aligned} \mu &= 1.4 / (1 + \frac{1.4 \sigma_n}{3.8}) \\ M &= 7.0 \\ X &= 0.3904 \cdot 10^{-5} \end{aligned} \right\} \dots\dots\dots (17)$$

for sandstone.

By using the strength relation $\sigma_1 = f(\sigma_3)$ for $\sigma_2 = \sigma_3$ the μ -value is found. Then by using the uniaxial and biaxial strength in (14) the value of M is found.

As the biaxial strength is not exactly determined we will have to make some adjustment for a better fit to the experimental points.

The calculations may now be made in two ways:

1. Calculating $\sigma_y = f(\sigma_x)$ at fracture when σ_z is constant.
2. Calculating the principal stress relation at fracture when the mean stress

$$\sigma_o = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \text{ is constant.}$$

These two methods are equivalent with cutting the space model of the strength perpendicular to the σ_z and space diagonal, respectively.

Fig. 5 and 6 give example of calculated strength curves for constant σ_z compared with experimental results. From the Figures we note that the calculated results are in good agreement with experimental values.

Fig. 7 and 8 show the calculated strength for constant mean stresses σ_o . The curves have shapes which show reasonable agreement with experimental result and are in agreement with what many scientist have expected (Mazanti 1967), (Sjibata and Karube 1965) and (Paul 1968).

CONCLUSIONS

The present theory has been able to, from a physical point of view, explain the variation in strength with different combinations of the principal stresses. By determining for example the uniaxial compressive strength, the internal friction coefficient $\mu(\sigma_n)$, and the variance in strength distribution M , a complete model of the strength of a material may be calculated. The calculated strength shows a result that many scientists have expected and experimental works have shown. For reasonable values of M the curves have no discontinuities or sharp corners. For great mean stresses where $\mu\sigma_n$ tends to a constant value τ_x the curves become Tresca hexagonals with rounded corners. For smaller σ_o the curves will have triangular forms.

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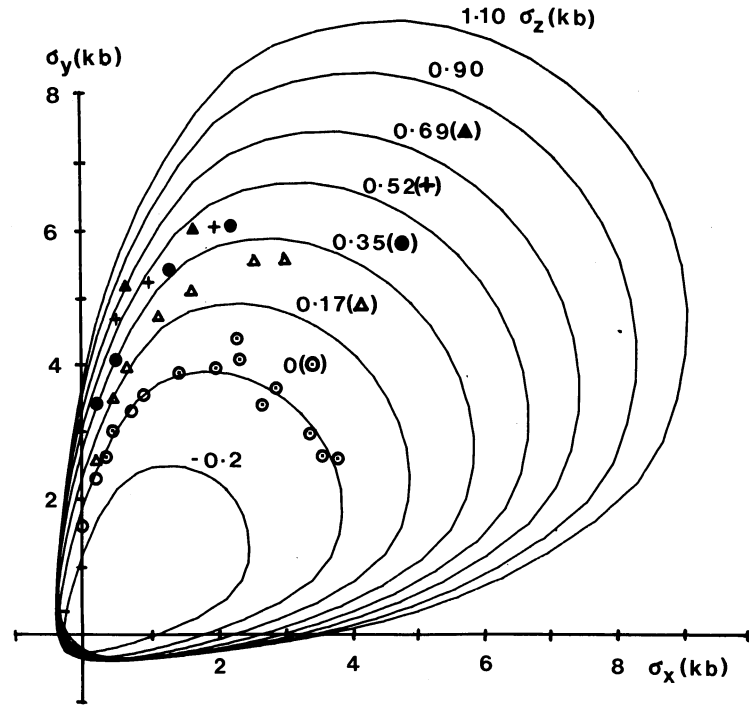


Fig. 5. Comparison between calculated strength and Hoskins' experimental values on trachyte. The symbols indicate different σ_z used by Hoskins corresponding to the calculated lines in the figure.

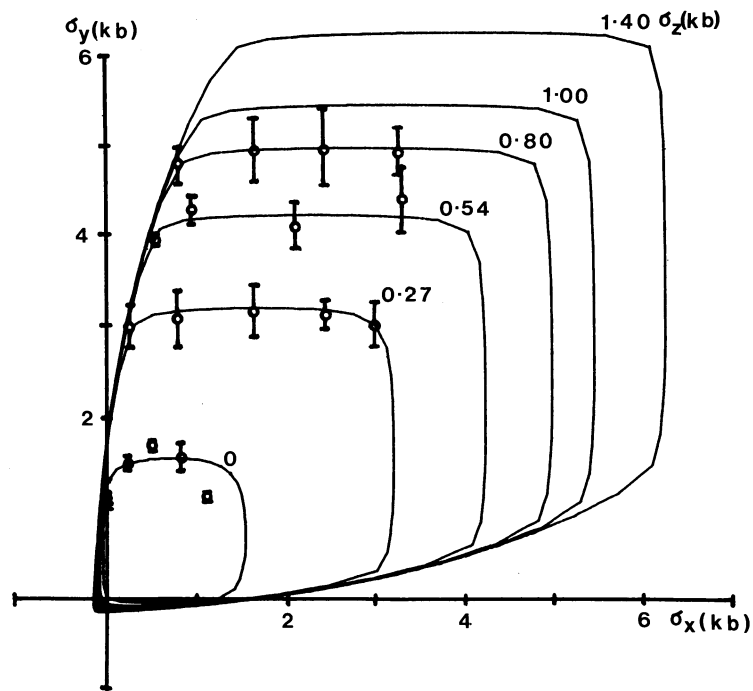


Fig. 6. Comparison between calculated strength and experimental values from Akai and Mori on sandstone.

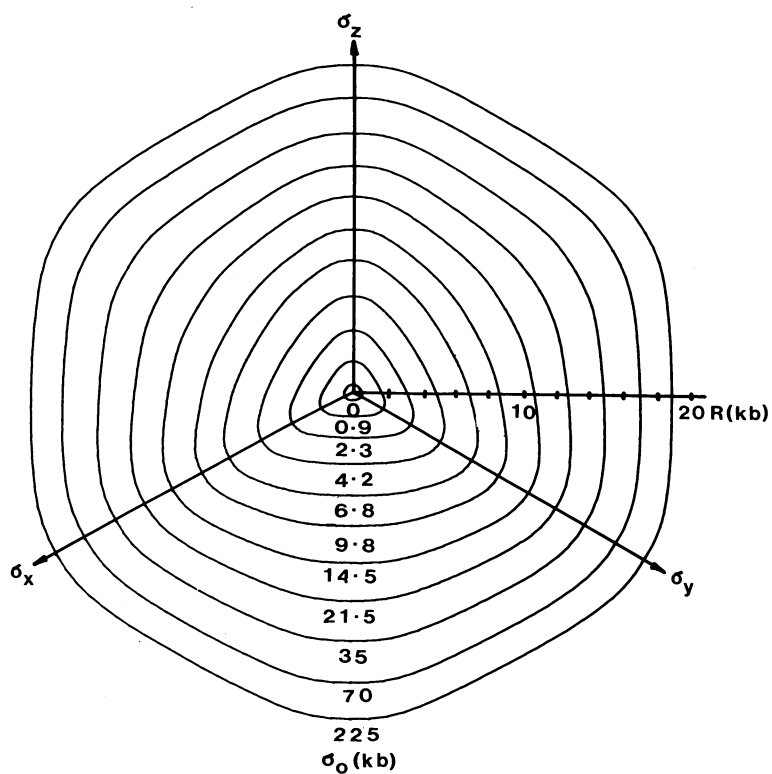


Fig. 7. Calculated strength criteria of trachyte for the deviatoric plane in the principal stress space at different mean stresses σ_0 . The R-scale shows the radius of the yield surface, and the σ -axes the direction of the principal stresses.

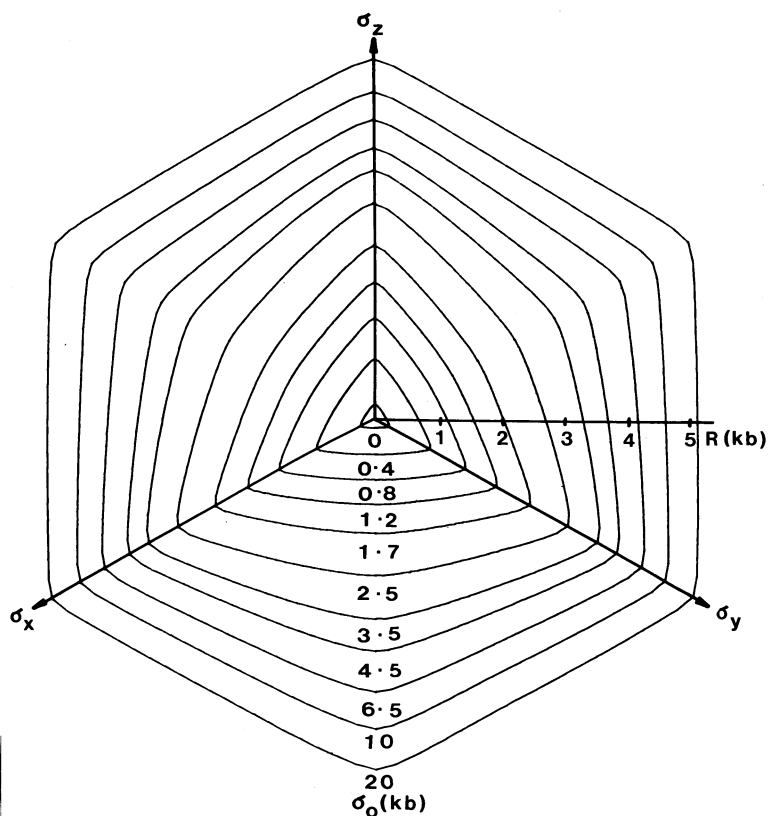


Fig. 8. Calculated strength criteria of sandstone for the deviatoric plane in the principal stress space at different mean stresses σ_0 . The R-scale shows the radius of the yield surface, and the σ -axes the direction of the principal stresses.

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APPENDIX

Deduction of the Relation Between Strength, μ and M

By the substitution of eqn (13) and

$$\alpha = \frac{\pi}{4} + \frac{\alpha_0}{2} + x \quad (\text{A.1})$$

in equation (12) we get

$$\left(\frac{\sigma_{20}}{\sigma_{10}}\right)^M = \frac{\int_{\cos(\frac{\pi}{4} + \frac{\alpha_0}{2} - x)}^{\cos(\frac{\pi}{4} + \frac{\alpha_0}{2} + x)} \sin(\frac{\pi}{4} + \frac{\alpha_0}{2} + x) dx}{\int_{\cos(\frac{\pi}{4} + \frac{\alpha_0}{2} - x)}^{\cos(\frac{\pi}{4} + \frac{\alpha_0}{2} + x)} \cos(\frac{\pi}{4} + \frac{\alpha_0}{2} + x) dx} \quad (\text{A.2})$$

where the integration is carried out for $x = -(\frac{\pi}{4} - \frac{\alpha_0}{2})$ to $(\frac{\pi}{4} - \frac{\alpha_0}{2})$

$$\left(\frac{\sigma_{20}}{\sigma_{10}}\right)^M = \frac{\int_{-M}^M [(\cos(\frac{\alpha_0}{2} + x) + \sin(\frac{\alpha_0}{2} + x))] dx}{\int_{-M}^M [(\cos(\frac{\alpha_0}{2} + x) - \sin(\frac{\alpha_0}{2} + x))] dx} \quad (\text{A.3})$$

$$= \frac{\int_{-M}^M (\cos \frac{\alpha_0}{2} \cos x - \sin \frac{\alpha_0}{2} \sin x + \sin \frac{\alpha_0}{2} \cos x + \cos \frac{\alpha_0}{2} \sin x) dx}{\int_{-M}^M (\cos \frac{\alpha_0}{2} \cos x - \sin \frac{\alpha_0}{2} \sin x - \sin \frac{\alpha_0}{2} \cos x - \cos \frac{\alpha_0}{2} \sin x) dx} \quad (\text{A.4})$$

The expression in the square brackets is an even function of x . Thus all integrals containing $\sin x$ are equal to zero and we get

$$\left(\frac{\sigma_{20}}{\sigma_{10}}\right)^M = \frac{\cos \frac{\alpha_0}{2} + \sin \frac{\alpha_0}{2}}{\cos \frac{\alpha_0}{2} - \sin \frac{\alpha_0}{2}} = \frac{1 + \sin \alpha_0}{\cos \alpha_0} \quad (\text{A.5})$$

Finally we get from (13) $\mu = \tan \alpha_0$ and (A.5) becomes

$$\left(\frac{\sigma_{20}}{\sigma_{10}}\right)^M = \sqrt{1 + \mu^2} + \mu \quad (\text{A.6})$$

Deduction of the above relation may also be done by using Guldin's rule.