

A Statistical Theory of the Polyaxial
Compressive Strength of Materials.

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Abstract

Weibull's statistical theory of strength, extended to compressive stresses, is used in calculating the influence of the intermediate principal stress on the strength. The effective shear stress

11 $\tau_e = |\tau_n| - \mu \sigma_n$ is calculated and integrated over the solid angle where $\tau_e > 0$, and the probability of rupture in polyaxial compression is calculated. The results are in good agreement with experimental results.

Introduction

The influence of the intermediate principal stress (σ_2) on the strength of brittle materials has long been a matter of discussion. Experimental work /3-7/ shows an increase in strength with increasing σ_2 , but up to this time no satisfactory explanation has been presented.

The most commonly used theory, that of Mohr-Coulomb, does not even show an effect of σ_2 . Wiebols and Cook /2/ recently presented an interesting theory, where an energy criterion is used, which reproduces the increase of strength with σ_2 found experimentally.

In the present work, the idea used by Weibull /1/ to explain the volume effect and polyaxial stress effect on the tensile strength of materials is applied. The theory is used in the compressive state of stress and an effective shear stress is used, defined in the same way as by Wiebols and Cook.

Statistical theory of strength

The conventional representation of Mohr-Coulomb's theory of strength is shown in fig. 1. Rupture will occur when τ_n in any point and at any angle (β) reaches the envelope (a).

If instead, we introduce an effective shear stress τ_e that appears over a certain angle we may state that the probability of rupture is a function of τ_e and the size of the angle.

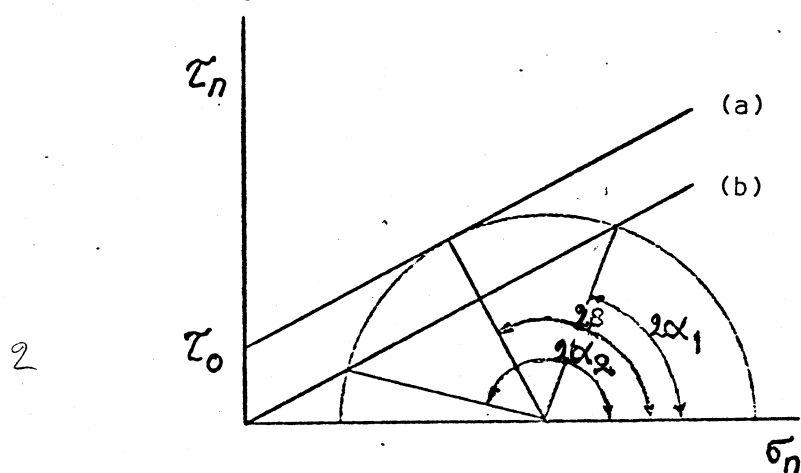


Fig. 1. Stress relations in the τ, σ plane.

If we lower the line (a) by the amount (τ_0) to the line (b), in fig. 1 we see that the shear stress (τ_n) in an angle $\alpha_2 - \alpha_1$ exceeds the frictional stress $(\mu \sigma_n)$. Let us define the effective shear stress $\tau_e^{*)} = \tau_n - \mu \sigma_n$ which is positive in that angle.

Shear- and normal stresses in polyaxial stress field

In the polyaxial stress state we can calculate the stresses in a plane, whose normal (OP fig. 2) has the direction cosines l, m and n to the σ_1, σ_2 and σ_3 axes, respectively. These are found to be

$$\sigma_n = l^2 \sigma_1 + m^2 \sigma_2 + n^2 \sigma_3 \quad (1)$$

and

$$\tau_n^2 = l^2 \sigma_1^2 + m^2 \sigma_2^2 + n^2 \sigma_3^2 - \sigma_n^2 \quad (2)$$

where

$$l^2 + m^2 + n^2 = 1 \quad (3)$$

By definition

$$\tau_e = \tau_n - \mu \sigma_n \quad (4)$$

From fig. 2 we get the relation

$$\begin{aligned} l &= \cos \alpha \cos \lambda \\ m &= \cos \alpha \sin \lambda \\ n &= \sin \alpha \end{aligned} \quad (5)$$

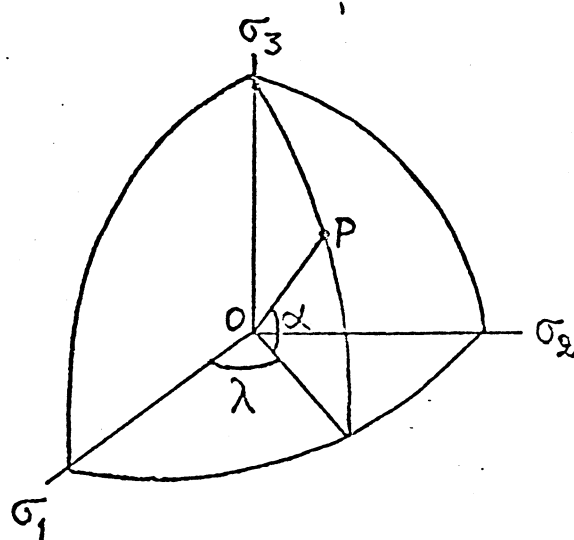


Fig. 2. Unit sphere with principal stress axes.

*) The effective shear stress τ_e , may have other expressions such as

$$\tau_e = \frac{|\tau_n|}{\tau_0 + \mu \sigma_n}, \text{ where the integration should be made over all}$$

directions.

By using (1) - (5) we can now calculate the solid angle (Ω) where $\tau_e > 0$. 3.

The result is shown in fig. 3 with $\sigma_1 = 8$ kb, $\sigma_3 = 1$ kb, and $\mu = 1$. In the shaded area of the angle no rupture can take place for any value of σ_2 between σ_1 and σ_3 .

From fig. 3 it may be seen that the solid angle at first decreases with increasing σ_2 , reaches a minimum, and then again increases when $\sigma_3 < \sigma_2 < \sigma_1$.

As in the plane case we may assume that the probability of rupture is a function of $\tau_e > 0$ and the solid angle Ω .

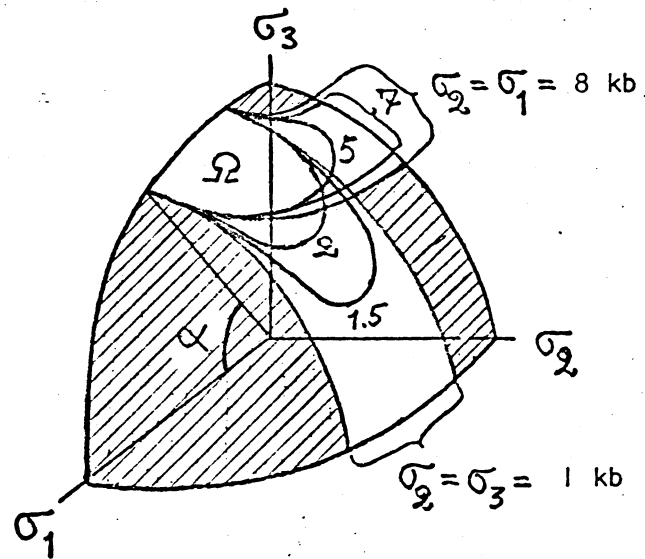


Fig. 3. Solid angles for $\tau_e > 0$ at different σ_2 when $\sigma_1 = 8$ kb, $\sigma_3 = 1$ kb and $\mu = 1$.

The probability function

Weibull has put the probability of rupture in tension in the form:

$$S(x) = 1 - e^{-kX} \quad (6)$$

where

$$X = \int \sigma^M d\Omega \quad (7)$$

and k and M are constants for a given material.

If we now put

$$X = \int \tau_e^M d\Omega \quad (8)$$

where the integration is carried out over the solid angle where $\tau_e > 0$, we may in the same way get the probability of rupture in the compressive state of stress. M and k may have other values than those found from tensile fracture.

By using experimental values for uniaxial compression say those for 50% rupture, we will find the value of kX from (6). If then μ and M are known, we can calculate the strength at any combination of the stresses by letting X be constant. This is easily done by using a computer.

Fig. 4 shows the variation in strength with σ_2 when $\sigma_3 = 0$, $\mu = 1$ and $M = 2$. For $M = 2$ the result coincides with that from Wiebols and Cook. We find that their treatment is a special case in the present general theory.

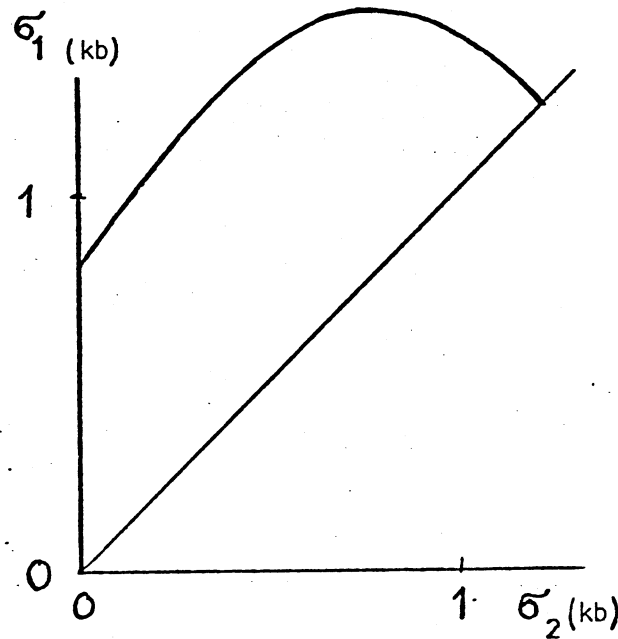


Fig. 4. Strength variation with σ_2 when $\sigma_3 = 0$, $\mu = 1$ and $M = 2$.

Relation between strength, μ and M

In the case when $\sigma_3 = m = 0$ and $\sigma_2 = \sigma_3$ or $\sigma_2 = \sigma_1$ we get from (1) - (4)

$$\begin{aligned}\sigma_n &= l^2 \sigma_1 \\ \tau_n &= l \cdot n \sigma_1\end{aligned}\tag{9}$$

and

$$\tau_e = \sigma_1 l(n - \mu l)$$

From fig. 3 it may be seen that this state of stress is held constant when rotating around either the σ_1 ($\sigma_2 = \sigma_3$) or σ_3 ($\sigma_2 = \sigma_1$) axis.

Using (5) and (9) this gives as $\lambda = 0$ ($m = 0$)

$$\tau_e = \sigma_1 \cos \alpha (\sin \alpha - \mu \cos \alpha) \quad (10) \quad 5.$$

where

$$\tan \alpha = \mu \text{ is constant}$$

When X is held constant as stated above (8) and (10) now give

$$\left(\frac{\sigma_{20}}{\sigma_{10}} \right)^M = \frac{\int_{\alpha_0}^{\pi/2} [\cos \alpha (\sin \alpha - \mu \cos \alpha)]^M \sin \alpha d\alpha}{\int_{\alpha_0}^{\pi/2} [\cos \alpha (\sin \alpha - \mu \cos \alpha)]^M \cos \alpha d\alpha} \quad (11)$$

where σ_{10} and σ_{20} are the uniaxial and biaxial compressive strengths, respectively.

It can be shown (See appendix) that (11) gives

$$\left(\frac{\sigma_{20}}{\sigma_{10}} \right)^M = \sqrt{1 + \mu^2} + \mu \quad (12)$$

This is a simple and useful relation for determination of one of the constants when the other three are known.

Fig. 5 shows the variation of strength for different M -values when $\mu = 1$. We see that the influence of σ_2 decreases with increasing M , and for $M = \infty$ we will have results in agreement with the classical Mohr-Coulomb theory.

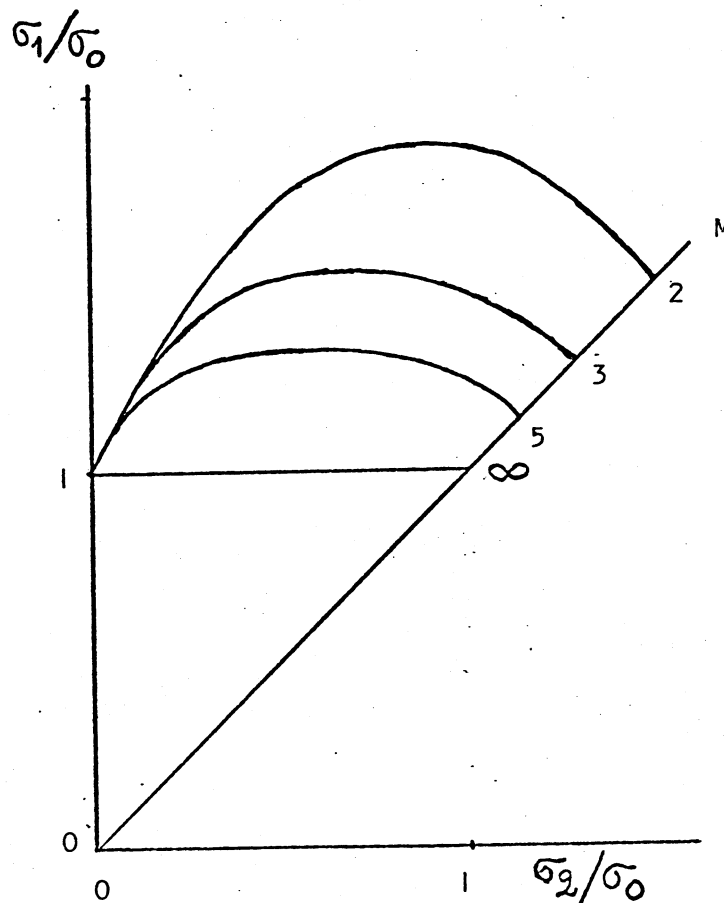


Fig. 5. Strength variation with σ_2 at different M when $\mu = 1$ and $\sigma_3 = 0$

The μ -value as a function of the normal stress

In the literature the μ -value is mostly considered independent of the normal stress. If μ is defined from the simple relation $\tau_f = \mu \sigma_n$, μ is not a constant at high normal stresses where τ_f increases only slowly with σ_n . In this region many expressions are given that all show a decreasing μ -value with increasing σ_n . Lundborg /9/ has used the relation

$$\mu = \mu_0 / (1 + \frac{\mu_0 \sigma_n}{\tau_x}) \quad (13)$$

which is derived from experimental results. μ_0 is a constant and τ_x the friction stress when $\sigma_n \rightarrow \infty$.

By using a computer, the expression (13) may easily be used instead of a constant μ -value.

Comparison between experimental and calculated strength

Hoskins /3/ and Akai and Mori /4/ among others have measured the strength of trachyte and sandstone in polyaxial state of stress. Hoskins' values are plotted in fig. 6. By using the strength relation $\sigma_1 = f(\sigma_3)$ when $\sigma_2 = \sigma_3$ the μ -value is found to be

$$\mu \approx 0.9 / (1 + \frac{0.9 \sigma_n}{12}) \quad (14)$$

and by using the uniaxial- and biaxial strength together with the relation (12) we get

$$M \approx 1.2 \quad (15)$$

Using these values we can calculate the strength, which is shown as full lines in fig. 6. Fig. 7 shows in the same way the comparison between calculated and experimental result from Akai and Mori on sandstone.

The calculated values are in good agreement with the experimental results.

Combined tensile and compressive stresses

If we extend the calculations to combined tensile and compressive stresses, assuming there are some friction stresses which still decrease when

$\sigma_n < 0$, we can calculate the strength also in this region. We may then assume that the calculation will hold when, according to Griffith /12/

$\sigma_1 + 3 \sigma_3 > 0$, and then let σ_3 be constant when σ_1 tends to zero. Under this assumption it is possible to find the ratio between the compressive and tensile strength. From the values of Hoskins in the compressive region we find, in this way, the ratio 9.85. It is also possible to let the friction stress be zero when $\sigma_n < 0$, where the above mentioned ratio becomes 7.40. More experimental results are, however, needed to prove the validity of the calculations in the case of combined stresses.

The calculations in this work were carried out on a H P table top calculator 9100 B.

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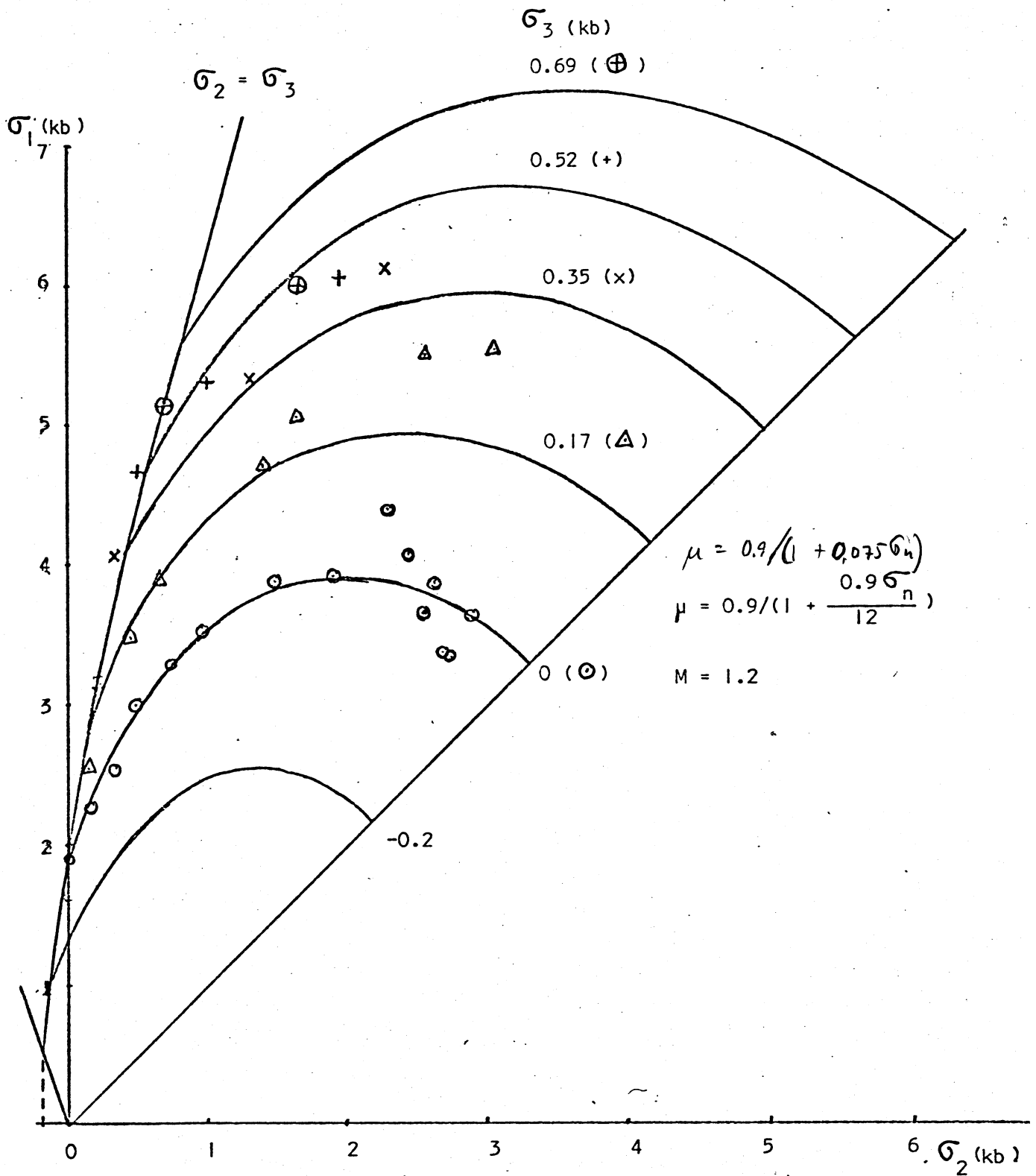


Fig. 6. Comparison between calculated strength and Hoskins' experimental values on trachyte. Different symbols referring to different σ_3 in Hoskins' measurements.

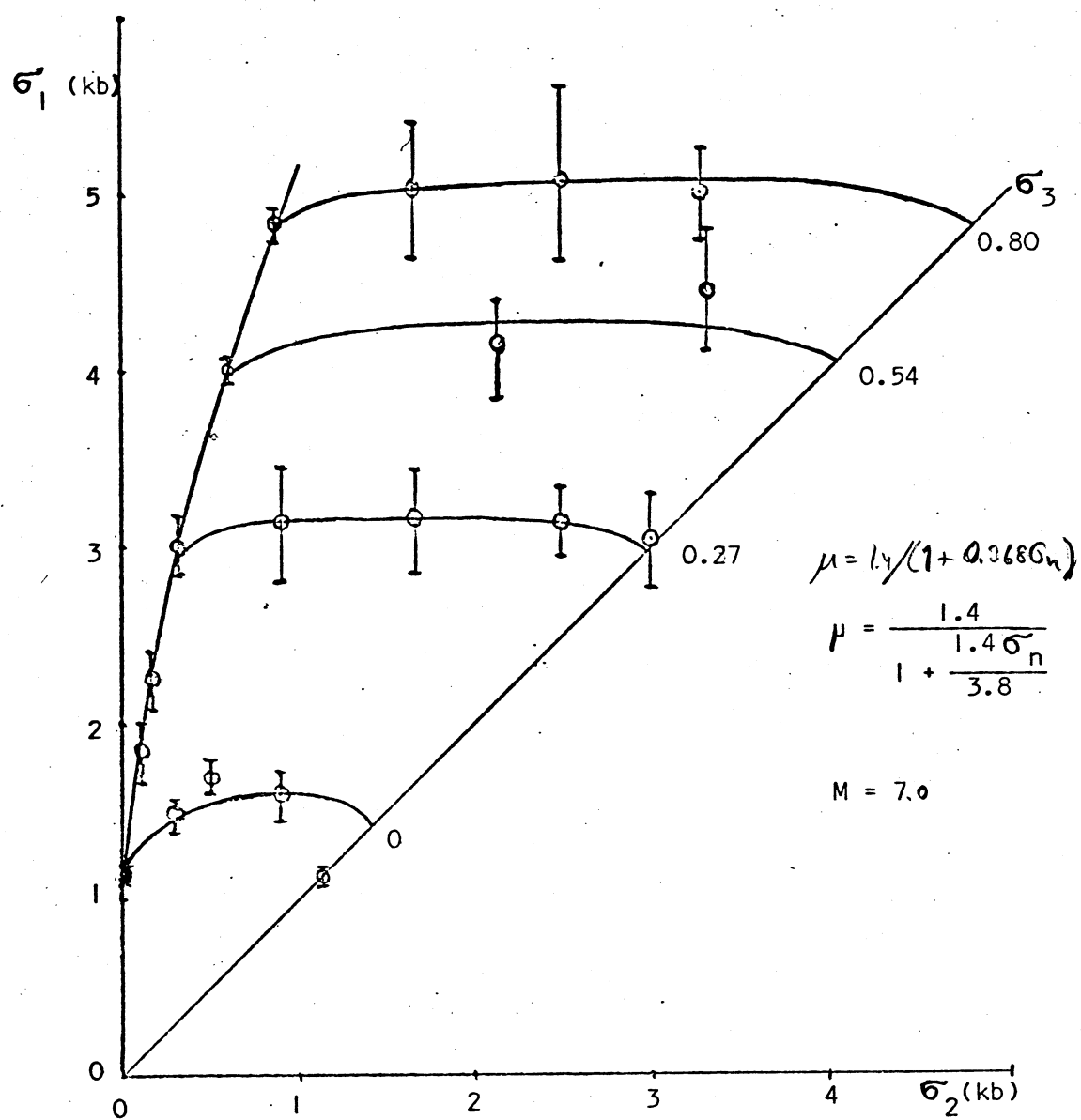


Fig. 7. Comparison between calculated strength and experimental values from Akai and Mori on sandstone.

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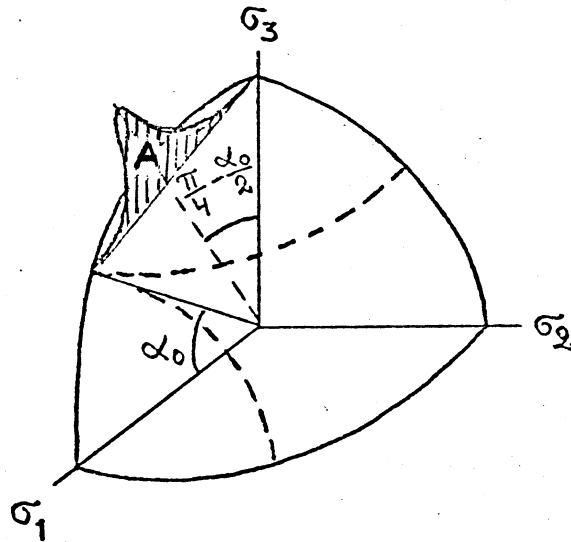


Fig. A 1. Unit sphere with principal stress axes as in fig. 2.

By using eqn. 10 and integrating the expression:

$$A = \int \tau_e^M d\alpha = \sigma_1^M \int_{\alpha_0}^{\pi/2} [\cos\alpha (\sin\alpha - \mu \cos\alpha)]^M d\alpha \quad (A1)$$

we find the shaded area A , in fig. A1

By the substitution

$$\alpha = \frac{\pi}{4} + \frac{\alpha_0}{2} + X \quad (A2)$$

eqn. (A1) gives

$$A = \sigma_1^M \int_{-(\frac{\pi}{4} - \frac{\alpha_0}{2})}^{\frac{\pi}{4} - \frac{\alpha_0}{2}} [\cos(\frac{\pi}{4} + \frac{\alpha_0}{2} + X) \cos(\frac{\pi}{4} + \frac{\alpha_0}{2} - X)]^M dX \quad (A3)$$

As the integrand is an even function of X , we find the center of gravity of the integrated area A , to be at $X = 0$, i.e., $\alpha = \frac{\pi}{4} + \frac{\alpha_0}{2}$.

Now by using Guldin's rule the integrals in (11) are simply obtained by rotating the area in fig. A1 around the σ_1 and σ_3 axes, respectively, so that (11) will be

$$\left(\frac{\sigma_{20}}{\sigma_{10}} \right)^M = \frac{\sin(\frac{\pi}{4} + \frac{\alpha_0}{2})}{\cos(\frac{\pi}{4} + \frac{\alpha_0}{2})} = \frac{1 + \sin\alpha_0}{\cos\alpha_0} \quad (A4)$$

Putting $\mu = \frac{\sin\alpha_0}{\cos\alpha_0}$ we finally get

$$\left(\frac{\sigma_{20}}{\sigma_{10}} \right)^M = \sqrt{1 + \mu^2} + \mu \quad (A5)$$